ASYMPTOTIC BEHAVIOUR FOR SOLUTIONS OF A NONLINEAR *p(x)*-TRIHARMONIC HYPERBOLIC TYPE EQUATION

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Abstract

In this work, we focus on the nonlinear *p(x)*-triharmonic equation without a source term with variable exponents

$$z_{tt} - \Delta^3_{p(x)} z + |z_t|^{q(x)-2} z_t = 0.$$

By utilizing by Komornik's lemma, we prove the asymptotic behaviour for the solution under suitable conditions on the variable exponents.

Keywords: *Asymptotic behaviour, triharmonic equation, variable exponents.*

1. INTRODUCTION

1.1 Model

In this article, we investigate the following p(x)-triharmonic problem:

$$\begin{cases} z_{tt} - \Delta^3_{p(x)} z + |z_t|^{q(x)-2} z_t = 0, \quad \Omega \times (0,T), \\ z(x,t) = \Delta z(x,t) = 0, \quad \partial\Omega \times (0,T), \\ z(x,0) = \varphi_1(x), \;\; z_t(x,0) = \varphi_2(x), \;\; x \in \Omega, \end{cases} \tag{1}$$

the $p(x)$-triharmonic $\Delta^3_{p(x)} z$ is the nonlinear differential operator defined by

$$\Delta^3_{p(x)} z = div\left(\Delta\left(|\nabla \Delta z|^{p(\cdot)-2} \nabla \Delta z\right)\right), \qquad \text{for all } z \in W^{3,p(\cdot)}(\Omega).$$

Ω is a bounded domain of R^n with a smooth boundary $\partial\Omega$, and the variable exponents $p(x)$ and $q(x)$ are given as measurable functions on Ω satisfying

$$2 \le p_1 \le p(x) \le p_2 \le q_1 \le q(x) \le q_2 \le p_1^* \tag{2}$$

with

$$\begin{cases} p_1 = essinf_{x\in\Omega} p(x), & p_2 = esssup_{x\in\Omega} p(x), \\ q_1 = essinf_{x\in\Omega} q(x), & q_2 = esssup_{x\in\Omega} q(x), \end{cases}$$

and

$$p_1^* = \begin{cases} \dfrac{np_1}{n-p_1} & if\ n > p_1 \\ +\infty & if\ n \le p_1, \end{cases}$$

and the log-Hölder continuity condition for some $\mathcal{A} > 0$ and any $0 < \delta < 1$,

$$|p(x) - p(y)| \le -\frac{\mathcal{A}}{ln|x-y|}, \quad for\ all\ x, y \in \Omega,\ with\ |x-y| < \delta. \tag{3}$$

1.2 Literature review

Liu [11] discussed the initial boundary value problem for a $p(x)$-fourth-order parabolic equation with nonstandard growth conditions

$$z_t + \Delta_{p(x)}^2 z = |z|^{q(x)-2} z.$$

He establish the local existence of weak solutions and derive the finite time blow up of solutions with non-positive initial energy.

Messaoudi et al. [14] investigate the following problem

$$z_{tt} - div\left(|\nabla z|^{r(\cdot)-2}\nabla z\right) + |z_t|^{m(\cdot)-2} z_t = 0$$

problem under suitable assumptions on the initial value, decay estimate was proved by using a lemma by Komornik.

Ferreira et al. [7] studied a nonlinear Petrovsky equation with a strong damping and the $p(x)$-biharmonic term as follows

$$z_{tt} + \Delta_{p(x)}^2 z - div\left(|\nabla z|^{r(\cdot)-2}\nabla z\right) + |z_t|^{m(\cdot)-2} z_t = f(x, t, z_t).$$

They used the the Faedo-Galerkin method to establish the existence of weak solution.

Get et al. [8] investigated the following equation:

$$\Delta\left(|\Delta z|^{p(x)-2}\Delta z\right) = \lambda V(x)\left(|z|^{q(x)-2} z\right).$$

They demonstrated the existence of a continuous family of eigenvalues, considering various cases of growth rates involved in the problem.

Belakhdar et al. [3] study the properties of the eigenvalue of the $p(x)$-triharmonic problems

$$-\Delta^3_{p\,(x)} z = \lambda V(x)\left(|z|^{q(x)-2} z\right),$$

under adequate conditions on the exponent functions p, q and the weight function $V_1(x)$.

The existence, nonexistence and decay of solutions was studied by many authors for the equation with variable exponents, see for instance [1, 2, 5, 7, 15, 16, 17, 18, 20, 21, 22, 23, 24, 26].

1.3 Main contribution

In this work, we prove the asymptotic behaviour for the problem (1). To the best of my knowledge, this is the first work of the hyperbolic type equation that includes a variable exponent triharmonic operator. Our study is motivated by Messaoudi et al. [14].

This work is organized as follows: Part 2 presents preliminary information regarding variable exponent Lebesgue and Sobolev spaces. Part 3 focuses on proving the decay solutions.

2 PRELIMINARIES

This part we give some preliminary facts and definitions about the Lebesgue spaces and Sobolev spaces with variable exponents (see [6, 10, 19]). Let $p : \Omega \longrightarrow [1, \infty]$ be a measurable function. The Lebesgue space with a variable exponents is defined by

$$L^{p(x)}(\Omega) = \left\{z\colon \Omega \longrightarrow R, \text{measurable in } \Omega \text{ and } \rho_{p(\cdot)}(\lambda z) < \infty, \text{for some } \lambda > 0\right\},$$

where

$$\rho_{p(\cdot)}(z) = \int_\Omega |z|^{\,p(x)}\, dx.$$

Luxembourg-type norm:

$$\|z\|_{p(\cdot)} = \left\{\lambda > 0\colon \int_\Omega \left|\frac{z(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space [10].

We next define the variable-exponent Sobolev space $W^{m,p(x)}(\Omega)$ as follows:

$$W^{m,p(x)}(\Omega) = \left\{z \in L^{p(\cdot)}(\Omega) \text{ such that } D^\alpha z \in L^{p(\cdot)}(\Omega), |\alpha| < m\right\}.$$

This is a Banach space.

Let us note that the space $W_0^{1,p(x)}(\Omega)$ has a different definition in the case of variable exponents.

However, under condition (3) both definitions are equivalent [4].

Lemma 1 [10] Let $\Omega \subset R^n$ be a bounded domain and $p(x)$ satisfies (3) condition, then

$\|z\|_{p(\cdot)} \leq C\|\nabla z\|_{p(\cdot)}$, for $z \in W_0^{1,p(\cdot)}(\Omega)$.

Lemma 2 [10] Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Suppose that $p : \Omega \longrightarrow (1, \infty)$ is a measurable function such that

$$2 \leq p_1 \leq p(x) \leq p_2 < \infty, \qquad \text{for a.e.,} \qquad x \in \Omega.$$

If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)\ in\ \overline{\Omega}$ with

$$p^*(x) = \begin{cases} \dfrac{np(x)}{n - p(x)} & \text{if } n > p_2, \\ +\infty & \text{if } n \leq p_2. \end{cases}$$

Then $W_0^{1,p(\cdot)} \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact embedding.

Lemma 3 [10] If

$$1 \leq q_1 := essinf_{x\in\Omega} q(x) \leq \ q_2 := esssup_{x\in\Omega} q(x) < \infty$$

then we have

$$min\left\{\|z\|_{q(\cdot)}^{q_1}, \|z\|_{q(\cdot)}^{q_2}\right\} \leq \rho_{q(\cdot)}(z) \leq max\left\{\|z\|_{q(\cdot)}^{q_1}, \|z\|_{q(\cdot)}^{q_2}\right\},$$

for any $z \in L^{q(\cdot)}(\Omega)$.

Lemma 4 [10] Suppose that $\mathrm{p}, \mathrm{s}, \mathrm{q}: \Omega \longrightarrow [1, \infty)$ be measurable functions defined on such that

$$\frac{1}{s(l)} = \frac{1}{p(l)} + \frac{1}{q(l)}, \text{for a.e. } l \in \Omega.$$

If $v_1 \in L^{p(\cdot)}(\Omega)$ and $v_2 \in L^{q(\cdot)}(\Omega)$, then $v_1 v_2 \in L^{s(\cdot)}(\Omega)$, with

$$\|v_1 v_2\|_{s(\cdot)} \leq \|v_1\|_{p(\cdot)} \|v_2\|_{q(\cdot)}.$$

3 ASYMPTOTIC BEHAVIOUR

In this part, we aim to demonstrate our main asymptotic behaviour. Firstly, we define the energy functional

$E(t) = \frac{1}{2}\|z_t\|^2 + \int_\Omega \frac{|\nabla\Delta z|^{p(x)}}{p(x)} dx.$

Lemma 5 Energy functional is non-increasing function.

Proof. Multiplying the equation of (1) by z_t integrating over Ω, using integration by parts, we obtain

$$E'(t) = -\int_\Omega |z_t|^{q(x)} dx \le 0. \qquad \square$$

We state the following theorem which can be obtained by the Faedo-Galerkin method and using the similar arguments as in [12] and [13].

Theorem 6 Let $(z_0, z_1) \in \left(W_0^{3,p(\cdot)}(\Omega), L^2(\Omega)\right)$ and p and q satisfies (2) and (3). Then problem (1) has a unique weak solution

$$\begin{cases} z \in L^\infty\left((0,T), W_0^{3,p(\cdot)}(\Omega)\right), \\ z_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{q(\cdot)}\left(\Omega \times (0,T)\right), \end{cases} \tag{5}$$

where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Lemma 7 [9] Let $E: R^+ \longrightarrow R^+$ be a non-increasing function. Assume that there exist $\sigma > 0$, $\omega > 0$ such that

$$\int_s^\infty E^{1+\tau}(t)dt \le \frac{1}{\omega} E^\tau(0) E^\tau(s), \quad \forall s > 0. \tag{6}$$

Then, $\forall t \ge 0$,

$$E(t) \le \begin{cases} \dfrac{cE(0)}{(1+t)^{\frac{1}{\tau}}}, & if \ \ \tau > 0 \\ cE(0)e^{-\omega t}, & if \ \ \tau = 0. \end{cases}$$

Lemma 8 [14] Let z be a solution of (1) and $0 \le E(t) \le E(0)$. Then, for some $C > 0$,

$\varrho_{p(x)}(\nabla z) \ge C\|\nabla z\|_{p_1}^{p_2}$.

By the above the Lemma, we have following inequality

$$\varrho_{p(x)}(\nabla\Delta z) \ge C\|\nabla\Delta z\|_{p_1}^{p_2}. \tag{7}$$

Theorem 9 Assuming that (2) and (3) hold. Then there exist two positive constants α, c independent of t and depend on $E(0)$ such that the energy $E(t)$ satisfies, $\ \ t \ge 0$,

$$E(t) \leq \begin{cases} \dfrac{c}{(1+t)^{\frac{2}{q_2-2}}}, & if \;\; q_2 > 2, \\ ce^{-\alpha t}, & if \;\; q(x) = 2. \end{cases}$$

Proof. Multiplying (1) by $z(t)E^r(t)$ and integrating over $\Omega \times (S,T)$, we get

$$\int_S^T E^r(t) \int_\Omega z(t)\left[z_{tt} - \Delta^3_{p(x)} z + |z_t|^{q(x)-2} z_t\right] dxdt = 0$$

which implies that

$$\int_S^T E^r(t) \int_\Omega \left[\left(z(t)z_t(t)\right)_t - |z_t(t)|^2 + |\nabla\Delta z(t)|^{p(x)} + z(t)|z_t(t)|^{q(x)-2} z_t(t)\right] dxdt = 0.$$

It follows that

$$\int_S^T E^r(t) \frac{d}{dt} \int_\Omega \left(z(t)z_t(t)\right) dxdt - \int_S^T E^r(t) \int_\Omega \left(|z_t(t)|^2 + |\nabla\Delta z(t)|^{p(x)}\right) dxdt$$
$$- 2 \int_S^T E^r(t) \int_\Omega |z_t(t)|^2 dxdt$$
$$+ \int_S^T E^r(t) \int_\Omega z(t)|z_t(t)|^{q(x)-2} z_t(t) dxdt = 0. \quad (8)$$

By using the definition of $E(t)$ and the relation

$$\frac{d}{dt}\left(E^r(t) \int_\Omega z(t)z_t(t)dx \right) = r\, E^{r-1}(t)E'(t) \int_\Omega \left(z(t)z_t(t)\right)dx$$
$$+ E^r(t) \frac{d}{dt} \int_\Omega \left(z(t)z_t(t)\right)dx.$$

Eq. (8) becomes

$$2\int_S^T E^{r+1}(t) \leq - \int_S^T \frac{d}{dt}\left(E^r(t) \int_\Omega z(t)z_t(t)\, dx \right) dt$$
$$+ r \int_S^T E^{r-1}(t)E'(t) \int_\Omega z(t)z_t(t)dxdt + 2\int_S^T E^r(t) \int_\Omega |z_t(t)|^2 dxdt$$
$$- \int_S^T E^r(t) \int_\Omega z(t)|z_t(t)|^{q(x)-2} z_t(t) dxdt.$$

Estimates:

i. $\left| - \int_S^T \frac{d}{dt}\left(E^r(t) \int_\Omega z(t)z_t(t)\, dx \right) dt \right|$

$$
\begin{aligned}
&= \left| E^r(s)\int_\Omega z(x,s)z_t(x,s)\,dx - E^r(T)\int_\Omega z(x,T)z_t(x,T)\,dx \right| \\
&\le E^r(s)\left[\tfrac{1}{2}\int_\Omega z^2(x,s)\,dx + \tfrac{1}{2}\int_\Omega z_t^2(x,s)\,dx\right] \\
&\quad + E^r(T)\left[\int_\Omega \frac{1}{2}z^2(x,T)\,dx + \frac{1}{2}\int_\Omega z_t^2(x,T)\,dx\right] \\
&\le E^r(s)\left[\tfrac{1}{2}c_*\int_\Omega |\nabla\Delta z(x,s)|^2\,dx + E(s)\right] \\
&\quad + E^r(T)\left[\frac{1}{2}c_*\int_\Omega |\nabla\Delta z(x,T)|^2\,dx + E(T)\right]
\end{aligned}
$$

here c_* is the embedding constant. So, we get

$$
\begin{aligned}
\left|-\int_S^T \frac{d}{dt}\left(E^r(t)\int_\Omega z(t)z_t(t)\,dx\right)dt\right| &\le E^r(s)\left[c\|\nabla\Delta z(x,s)\|_{p_1}^2 + E(s)\right] \\
&\quad + E^r(T)\left[c\|\nabla\Delta z(x,s)\|_{p_1}^2 + E(T)\right] \\
&\le E^{r+1}(s) + cE^r(s)\left(\|\nabla\Delta z(s)\|_{p_1}^{p_2}\right)^{\frac{2}{p_2}} \\
&\le E^{r+1}(T) + cE^r(T)\left(\|\nabla\Delta z(T)\|_{p_1}^{p_2}\right)^{\frac{2}{p_2}}
\end{aligned}
$$

here c is a positive constant. Then we use (7) and recall that $E(t)$ is non-increasing to get

$$
\begin{aligned}
\left|-\int_S^T \frac{d}{dt}\left(E^r(t)\int_\Omega z(t)z_t(t)\,dx\right)dt\right| &\le E^{r+1}(s) + cE^r(s)\left(\varrho_{p(x)}\left(\nabla\Delta z(s)\right)\right)^{\frac{2}{p_2}} \\
&\quad + E^{r+1}(T) + cE^r(T)\left(\varrho_{p(x)}\left(\nabla\Delta z(T)\right)\right)^{\frac{2}{p_2}} \\
&\le E^{r+1}(s) + c\left(E(s)\right)^{r+\frac{2}{p_2}} \\
&\quad + E^{r+1}(T) + c\left(E(T)\right)^{r+\frac{2}{p_2}} \\
&\le E^{r+1}(s) + c\left(E(s)\right)^{r+\frac{2}{p_2}}. \qquad (10)
\end{aligned}
$$

ii.

$$
\begin{aligned}
\left|r\int_S^T E^{r-1}(t)E'(t)\int_\Omega z(t)z_t(t)dxdt\right| &\le -c\int_S^T E^{r-1}(t)E'(t)\left[E(t) + cE(t)^{\frac{2}{p_2}}\right] \\
&\le -c\left[\int_S^T E^r(t)E'(t)dt + \int_S^T \left(E(t)\right)^{r+\frac{2}{p_2}}E'(t)dt\right] \\
&\le c\left[E^r(s) + \left(E(s)\right)^{r+\frac{2}{p_2}}\right] \qquad (11)
\end{aligned}
$$

For the third term of the right-hand side of (9), we establish, as in [9],

$\Omega_+ = \{x \in \Omega: |z_t(x,t)| \geq 1\}$ and $\Omega_- = \{x \in \Omega: |z_t(x,t)| < 1\}$

and utilizing Hölder's inequality and Young's inequality and (2) as follows

iii. $$\left|2\int_S^T E^r(t)\int_\Omega |z_t(t)|^2 dxdt\right| = \left|2\int_S^T E^r(t)\left[\int_{\Omega_-} z_t^2 dx + \int_{\Omega_+} z_t^2 dx\right]\right|$$

$$\leq c\int_S^T E^r(t)\left[\left(\int_{\Omega_-}|z_t|^{q_2}dx\right)^{\frac{2}{q_2}} + \left(\int_{\Omega_+}|z_t|^{q_1}dx\right)^{\frac{2}{q_1}}\right]$$

$$\leq c\int_S^T E^r(t)\left[\left(\int_{\Omega}|z_t|^{q_2}dx\right)^{\frac{2}{q_2}} + \left(\int_{\Omega}|z_t|^{q_1}dx\right)^{\frac{2}{q_1}}\right]$$

$$\leq c\int_S^T E^r(t)\left(-E'^{(t)}\right)^{\frac{2}{q_2}}dt + c\int_S^T E^r(t)\left(-E'(t)\right)^{\frac{2}{q_1}}\mathrm{dt}$$

$$\leq c\varepsilon\int_S^T E^{r+1}(t)\,dt + c_\varepsilon\int_S^T\left(-E'(t)\right)^{\frac{2(r+1)}{q_2}}dt$$

$$+c\varepsilon\int_S^T\left(E(t)\right)^{\frac{rq_1}{(q_1-2)}}dt + c_\varepsilon\int_S^T \left(-E'(t)\right)dt$$

where $c_\varepsilon = \frac{1}{r+1}\left(\frac{\varepsilon(r+1)}{r}\right)^{-r}$. Choose r such that $r = \frac{q_2}{2} - 1$ and notice that $\frac{rq_1}{(q_1-2)} = r + 1 + \frac{q_2-q_1}{q_1-2}$. In the rest of the proof, we will consider two cases: $q_1 > 2$ and $q_1 = 2$.

- If $q_1 > 2$, we get

$$\left|2\int_S^T E^r(t)\int_\Omega |z_t(t)|^2 dxdt\right| \leq c\varepsilon\int_S^T E^{r+1}(t)\,dt$$

$$+c\varepsilon\left(E(0)\right)^{\frac{q_2-q_1}{q_1-2}}\int_S^T E^{r+1}(t)\,dt + c_\varepsilon E(s)$$

$$\leq \check{c}\varepsilon\int_S^T E^{r+1}(t)\,dt + c_\varepsilon E(s), \tag{12}$$

where $\check{c}$ is a positive constant independent of ε.

- If $q_1 = 2$, we get

$$\left|2\int_S^T E^r(t)\int_\Omega |z_t(t)|^2 dxdt\right| = \left|2\int_S^T E^r(t)\left[\int_{\Omega_-} z_t^2 dx + \int_{\Omega_+} z_t^2 dx\right]\right|$$

$$\leq c\int_S^T E^r(t)\left[\left(\int_{\Omega_-}|z_t|^{q_2}dx\right)^{\frac{2}{q_2}} + \int_{\Omega_+}|z_t|^{q(x)}dx\right]$$

$$\leq c \int_S^T E^r(t)\left[\left(\int_\Omega |z_t|^{q(x)}dx\right)^{\frac{2}{q_2}} + \int_\Omega |z_t|^{q(x)}dx\right]$$

$$\leq c \int_S^T E^r(t)\left(-E'^{(t)}\right)^{\frac{2}{q_2}}dt + c\int_S^T E^r(t)\left(-E'(t)\right)\mathrm{dt}$$

$$\leq c\varepsilon \int_S^T E^{r+1}(t)\,dt + c_\varepsilon \int_S^T \left(-E'(t)\right)^{\frac{2(r+1)}{q_2}}dt + cE^{r+1}(t).$$

Choose r such that $r = \frac{q_2}{2} - 1$. Hence, we get

$$\left|2\int_S^T E^r(t)\int_\Omega |z_t(t)|^2 dxdt\right| \leq c\varepsilon \int_S^T E^{r+1}(t)\,dt + c_\varepsilon E(s) + cE^{r+1}(t)$$
$$\leq c\varepsilon \int_S^T E^{r+1}(t)\,dt + \left(c_\varepsilon + cE^r(0)\right)E(s)$$
$$\leq c\varepsilon \int_S^T E^{r+1}(t)\,dt + \check{c}_\varepsilon E(s),$$

where $\check{c} = c_\varepsilon + cE^r(s)$.

Since $r = \frac{q_2}{2} - 1$ and $q_2 \geq p_2$, then $r + \frac{2}{p_2} - 1 \geq 0$, and, consequently, the estimates (10) and (11) become, respectively,

$$\left|-\int_S^T \frac{d}{dt}\left(E^r(t)\int_\Omega z(t)z_t(t)\,dx\right)dt\right| \leq E^{r+1}(s) + c\left(E(s)\right)^{r+\frac{2}{p_2}}$$
$$\leq \left[E^r(0) + c\left(E(0)\right)^{r+\frac{2}{p_2}-1}\right]E(s)$$
$$\leq \check{c}E(s). \qquad (13)$$

$$\left|r\int_S^T E^{r-1}(t)E'(t)\int_\Omega z(t)z_t(t)dxdt\right| \leq E^{r+1}(s) + c\left(E(s)\right)^{r+\frac{2}{p_2}}$$
$$\leq \left[E^r(0) + c\left(E(0)\right)^{r+\frac{2}{p_2}-1}\right]E(s)$$
$$\leq \check{c}E(s). \qquad (14)$$

For the last term of (9), we apply Young's inequality with $\mathrm{r(x)} = \frac{q(x)}{q(x)-1}$, $\mathrm{r(x)} = q'(x)$, so for a.e. $\mathrm{x} \in \Omega$, we obtain

$|z_t|^{q(x)-1}|z| \leq |z_t|^{q(x)} + c_\varepsilon |z|^{q(x)}$,

here

$c_\varepsilon(x) = \varepsilon^{1-q(x)}\left(q(x)\right)^{-q(x)}(q(x)-1)^{q(x)-1}$.

Therefore,

$$\left|\int_S^T E^r(t)\int_\Omega z|z_t|^{q(x)-1}dxdt\right|$$
$$\leq \int_S^T E^r(t)\int_\Omega \varepsilon|z|^{q(x)}dxdt + \int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt$$
$$\leq \varepsilon\int_S^T E^r(t)\left[\int_\Omega |z|^{q_1} + \int_\Omega |z|^{q_2}dxdt\right]z + \int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt$$

By utilizing the embedding, we get

$$\left|\int_S^T E^r(t)\int_\Omega z|z_t|^{q(x)-1}dxdt\right|$$
$$\leq \varepsilon\int_S^T E^r(t)\left[c_1|\nabla\Delta z|_{q_1}^{q_1} + c_2|\nabla\Delta z|_{q_1}^{q_2}\right] + \int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt,$$

where $c_1, c_1 > 0$ constants independent of ε. We recall (4) and (7) to obtain

$$\left|\int_S^T E^r(t)\int_\Omega z|z_t|^{q(x)-1}dxdt\right| \leq \varepsilon\int_S^T E^r(t)\left[c_1\left(\varrho_{p(x)}(\nabla\Delta z)\right)^{\frac{q_1}{p_2}} + c_2\left(\varrho_{p(x)}(\nabla\Delta z)\right)^{\frac{q_2}{p_2}}\right]$$
$$+\int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt,$$
$$\leq \varepsilon c_3\int_S^T E^{r+1}(t)\left(E(t)\right)^{\frac{q_1}{p_2}-1} + \varepsilon c_4\int_S^T E^{r+1}(t)\left(E(t)\right)^{\frac{q_2}{p_2}-1}$$
$$+\int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt$$
$$\leq \varepsilon c_5\left[\left(E(0)\right)^{\frac{q_1}{p_2}-1} + \left(E(0)\right)^{\frac{q_2}{p_2}-1}\right]\int_S^T E^{r+1}(t)$$
$$+\int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt \qquad (15)$$

$c_3, c_4, c_5 > 0$ constants independent of ε. Hence, a combination of (9)-(15) results in

$$2\int_S^T E^{r+1}(t) \leq c\varepsilon\left(\left(1+E(0)\right)^{\frac{q_1}{p_2}-1}\left(E(0)\right)^{\frac{q_2}{p_2}-1}\right) + \int_S^T E^{r+1}(t)\,dt$$
$$+ c_\varepsilon E(s)\ \int_S^T E^r(t)\int_\Omega c_\varepsilon(x)|z_t|^{q(x)}dxdt \qquad (16)$$

Choose ε enough that

$$c\varepsilon\left(\left(1+E(0)\right)^{\frac{q_1}{p_2}-1}\left(E(0)\right)^{\frac{q_2}{p_2}-1}\right)<1.$$

Once ε is fixed, then $c_\varepsilon(x)\leq K$ since $q(x)$ is bounded. Thus, we obtain

$$\begin{aligned}\int_S^T E^{r+1}(t) &\leq cE(s)+K\int_S^T E^r(t)\int_\Omega |z_t|^{q(x)}dxdt\\ &\leq cE(s)-K\int_S^T E^r(t)\ E'(t)dt\\ &\leq cE(s)-\frac{K}{r+1}\left(E^{r+1}(s)-E^{r+1}(T)\right)\\ &\leq c\left(E(s)+E^{r+1}(s)\right)\\ &\leq c\left(1+E^r(0)\right)E(s)=\check{c}E(s),\ \forall T>s>0.\end{aligned}$$

By taking $T\longrightarrow\infty$, we get

$$\int_S^T E^{r+1}(t)\leq\check{c}E(s),$$

Hence, Komornik's Lemma (with $\sigma=\frac{q_2}{2}-1$) implies the desired result. □

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