AN INVERSE SOURCE HYPERBOLIC PROBLEM WITH CAUCHY CONDITIONS AS A GENERALIZED PROBLEM OF MOMENTS

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Abstract

We consider the problem of finding a pair of functions $p(x)$ and $w(x,t)$ that satisfy the equation $w_{tt}(x,t) = w_{xx}(x,t) + f(t)p(x)$ under Cauchy boundary conditions. We will see that an approximate solution can be found using the techniques of generalized inverse problem of moments and find dimensions for the error of the estimated solution.

Keywords: *generalized moment problem; integral equations; hiperbolic equation, inverse source.*

INTRODUCTION

We want to find $p(x)$ and $w(x,t)$ such that

$$w_{tt}(x,t) = w_{xx}(x,t) + f(t)p(x)$$

under the Cauchy conditions using the problema generalized moments techniques.

Where $f(t) \in C^1(0,\infty)$ is known, and $p(x) \epsilon L^2(a_1, b_1)$.

Inverse source problems have significant applications in many scientific áreas. This paper studies an inverse hyperbolic problem. There are numerous works in the literatura that deal with this topic, for example, in [1] it is investigated the solvability of the inverse problem of finding a solution and an unknown coefficient in a pseudohyperbolic equation known as the Klein-Gordon equation. In [2] it is studied a new numerical method to solve an inverse source problem for general hyperbolic equations. This is the problem of reconstructing sources from the lateral Cauchy data of the wave field on the boundary of a domain. The paper [3] is concerned with inverse acoustic source problems in an unbounded domain with dynamical boundary surface data of Dirichlet kind. The measurement data are taken at a surface far away from the source support. The article [4] is concerned with two inverse problems on determining moving source profile functions in evolution equations with a derivative order $\alpha \in (0, 2]$ in time. In the first problem, the sources are supposed to move along known straight lines, and we suitably choose partial interior observation data in finite time. In [5] the main aim of this paper is to solve an inverse source problem for a general nonlinear hyperbolic equation. Combining the quasi-reversibility method and a suitable Carleman weight function, it defines a map of which fixed point is the solution to the inverse problem. In paper [6] we study the inverse problem of identifying a source or an initial state in a time fractional diffusion equation from the knowledge of a single boundary measurement. The paper [7] studies an inverse hyperbolic problem for the wave equation with dynamic boundary conditions. It consists of determining some forcing terms from the final overdetermination of the displacement. In [8] it develops an efficient and convergent numerical method for solving the inverse problem of determining the potential of nonlinear hyperbolic equations from lateral Cauchy data. In [9] it is estudied an early-warning inverse source problem for the elastogravitational equations. It consists of a mixed hyperbolic-elliptic system of partial differential equations describing elastic wave displacement and gravity perturbations produced by a source in a homogeneous bounded medium. In [10] is considered a transmission wave equation in N embedded domains with multiple interfaces of discontinuous coefcients in R^2. It is estudied the global stability in determining the source term from a one-measurement data of wavefield velocity in a subboundary over a time interval. It is tested the stability estimate for this inverse source problem by a combination of the local hyperbolic/elliptic Carleman estimates and the Fourier-Bros-Iagolniter transformation. In [11] and [12] is considered the inverse hyperbolic problem of recovering coefficients.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. The interest is not to compare with the existing methods, but to present a different method.

The generalized moments problem [13, 14, 15, 16, 17] is to find a function $f(x)$about a domain $\Omega \subset R^d$ that satisfies the sequence of equations

$$\mu_i = \int_\Omega g_i(x) f(x)dx \quad i \epsilon N ------------(1)$$

where N is the set of the natural numbers, $\left(g_i(x)\right)$ is a given sequence of functions in $L^2(\Omega)$ linearly independent known and the succession of real numbers μ_i are known data.
The problem of Hausdorff moments [15, 17], is to find a function $f(x)$en (a, b) such that

$$\mu_i = \int_a^b x^i f(x)dx \quad i \epsilon N --------(2)$$

In this case $g_i(x) = x^i$ with i belonging to the set N. If the integration interval is $(0, \infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty, \infty)$ we have the problem of Hamburger moments [15, 17]. The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [15, 16]. There are several methods to build regularized solutions. One of them is the truncated expansion method [16]. This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i = 1,2, \dots, n ---------- (3)$$

where it is considered as approximate solution of $f(x)$ to

$$p_n(x) = \sum_{i=0}^{n} \lambda_i \, \phi_i(x)$$

and the functions $\phi_i(x)_{i=0,1,\dots,n}$ result of orthonormalize $\langle g_1, g_2, \dots, g_n \rangle$ being λ_i the coefficients based on the data μ_i . In the subspace generated by $\langle g_1, g_2, \dots, g_n \rangle$ the solution is stable. If $n \in N$ is chosen in an appropriate way then the solution of (3) it approaches the solution of the original problem (1).

In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution ([16] p.p. 19 - 30).

ARTICLE ORGANIZATION

We want to find $p(x)$ and $w(x,t)$ such that

$$w_{tt}(x,t) = w_{xx}(x,t) + f(t)p(x)$$

under the Cauchy conditions using the problema generalized moments techniques

We will see that the problem can be solved in three steps.

In steps one and two we found an approximation for $w(x,t)$. Then in a third step an approximate solution is found for $p(x)$.

The next section describes the first step. The section that follows it is explained how the generalized moment problem is solved with the truncated expansion method and explains the second step where an approximation for $w(x,t)$ is found. Then an approximate solution is found for $p(x)$. Finally the numerical examples and the conclusions.

APPROXIMATION OF $w(x,t)$.

You want to find $w(x,t)$ such that

$$w_{tt}(x,t) = w_{xx}(x,t) + f(t)p(x) ------------ (4)$$

about a domain $E = (a_1, b_1) \times (0, \infty)$ with the conditions

$$w(x,0) = k(x) \quad a_1 \leq x \leq b_1$$

$$w_t(x,0) = g(x) \quad a_1 \leq x \leq b_1$$

$$w(b_1,t) = s_1(t) \quad w(a_1,t) = s_2(t) \quad t > 0$$

$$w_x(b_1,t) = h_1(t) \quad w_x(a_1,t) = h_2(t) \quad t > 0$$

We take as an auxiliary function

$$u(m,r,x,t) = e^{-m\left(\frac{x}{b1}\right)} e^{-r(t)}$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w_t)$$

We have

$$div(F^*) = w_{xx}(x,t) - w_{tt}(x,t) = -f(t)p(x) ---------(5)$$

Moreover, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, then

$$\iint_E u\, div(F^*) dA = \iint_E div(uF^*) dA - \iint_E F^*.\nabla u dA ---------- (6)$$

where $\nabla u = (u_x, u_t)$

Besides that

$$\iint_E div(uF^*) dA = \iint_E (uw_x)_x - (uw_t)_t dA =$$

$$= \iint_E u\, div(F^*) dA + \iint_E (u_x w_x - u_t w_t) dA --------------- (7)$$

Then from (6) y (7):

$$\iint_E (u_x w_x - u_t w_t) dA = \iint_E F^*.\nabla u dA -------------- (8)$$

On the other hand, it can be proven, after several calculations that, integrating by parts:

$$\iint_E F^* . \nabla u dA =$$
$$A(m,r) + B(m,r) - \iint_E u\, w\left(\left(\frac{m}{b_1}\right)^2 - (r)^2\right) dA = \varphi(m,r) - - - - - - - - - - (9)$$

with

$$A(m,r) = \int_0^{\infty} \left(-\frac{m}{b_1}\right) u(m,r,b_1,t) w(b_1,t) - \left(-\frac{m}{b_1}\right) u(m,r,a_1,t) w(a_1,t) dt$$
$$B(m,r) = \int_{a_1}^{b_1} (-r) u(m,r,x,b_2) w(x,b_2) - (-r) u(m,r,x,a_2) w(x,a_2) dx$$

If $\frac{m}{b_1} = r$, instead of (8) y (9):

$$\iint_E (-r) u\, w_x - (-r) w_t u dA = \varphi(rb_1,r)$$
$$\therefore \iint_E u(-w_x + w_t) dA = \frac{\varphi(rb_1,r)}{r}$$

with

$$\frac{\varphi(rb_1,r)}{r} =$$
$$= \int_0^{\infty} -r\, u(rb_1,r,b_1,t) w(b_1,t) + r u(rb_1,r,a_1,t) w(a_1,t) dt +$$
$$+ \int_{a_1}^{b_1} -u(rb_1,r,x,b_2)(-rb_1) w(x,b_2) - u(rb_1,r,x,a_2)(-rb_1) w(x,a_2) dx$$

We note $\varphi_1(r) = \frac{\varphi(rb_1,r)}{r}$, then

$$\iint_E u(-w_x + w_t) dA = \varphi_1(r) - - - - - - - - - - - - - (10)$$

To solve this integral equation we take a base $\psi_i(r) = r^i e^{-r} \quad i = 0,1,2,\dots,n$ of $L^2(E)$. Then we multiply both members of (10) by $\psi_i(r) = r^i e^{-r}$ and we integrate with respect to r, we obtain

$$\iint_E H_i(x,t)(-w_x + w_t) dA = \int_0^{\infty} \varphi_1(r)\psi_i(r) dr = \mu_i \quad i = 0,1,2,\dots,n - - - - (11)$$

where $H_i(x,t) = \int_0^{\infty} u(-rb_1, r, x, t)\psi_i(r)dr$.
We can interpret (11) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation $p_{1n}(x,t)$ for $-w_x + w_t$.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [17] and generalized in [19] and [20] to find an approximation $p_{1n}(x,t)$ of $-w_x + w_t$ for the corresponding finite problem with $i = 0,1,2,\dots,n$, where n is the number of moments μ_i .
We consider the basis $\phi_i(x,t)$ $\quad i = 0,1,2,\dots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t)$ $\quad i = 0,1,2,\dots,n$.
We approximate the solution $-w_x + w_t$ with [13] and generalized in [19] y [20]:

$$p_{1n}(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{where} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\, \mu_j \qquad i = 0,1,2,\dots,n$$

and the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) | \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) . \|\phi_i(x,t)\|^{-1} \quad 1 < i \leq n\,; 1 \leq j < i\,.$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0,1,\dots,n.$$

The proof of the following theorem is in [19] and [20].
In [20] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [18] the demonstration is made for the one-dimensional case.
This Theorem gives a measure about the accuracy of the approximation.

Theorem

Let $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2\big((a_1,b_1) \times (a_2,b_2)\big)$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n} \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} H_i(x,t) f(x,t) dx\, dt - \mu_i \right|^2 \leq \varepsilon^2$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}((b_1-a_1)^2 f_x^2+(b_2-a_2)^2 f_t^2)dx\,dt \leq M^2 ------------ (12)$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f(x,t)|^2\,dtdx \leq min_i\left\{\|C^T C\|\varepsilon^2+\frac{M^2}{8(n+1)^2};i=0,1,\dots,n\right\}$$

where C it is a triangular matrix with elements C_{ij}, $1<i\leq n;\ 1\leq j<i$
and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_{1n}(x,t)-f(x,t)|^2\,dxdt \leq \|CC^T\|\varepsilon^2+\frac{M^2}{8(n+1)^2} ------------ (13)$$

If b_2 it is not finite then (12}) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}(xf_x^2+tf_t^2)dx\,dt \leq M^2$$

And it must be fulfilled that $t^i f(x,t)\longrightarrow 0$ if $t\longrightarrow\infty$ $\forall i\epsilon N$.

So we have an equation in first order partial derivatives of the form

$$-w_x(x,t)+w_t(x,t)=p_{1n}(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t)+A_2(x,t)w_t(x,t)=p_{1n}(x,t)$$

where $A_1(x,t)=-$ and $A_2(x,t)=1$.
It is resolved as in [18], that is, we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{0}^{\infty}K(m,r,x,t)w(x,t)dtdx=\varphi_2(m,r) -------- (14)$$

With $K(m,r,x,t)=u(m,r,x,t)\left(\frac{m}{b_1}-r\right)$ where now it is taken as an auxiliary function

$$u(m,r,x,t)=e^{-m\left(\frac{x}{b_1}\right)}e^{-r(t)}$$

and

$$\varphi_2(m,r) = \int_{a_1}^{b_1} u(m,r,x,b_2)w(x,b_2) - u(m,r,x,a_2)w(x,a_2)dx -$$

$$-\int_0^{\infty} u(m,r,b_1,t)\, w(b_1,t) - u(m,r,a_1,t)w(a_1,t)dt - \int_0^{\infty}\int_{a_1}^{b_1} p_{1n}(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)} \qquad i = 0,1,\dots,n_1 \quad j = 0,1,2,\dots,n_2$$

and we multiply both members of (14) by $\psi_{ij}(m,r)$ and we integrate with respect to m and r

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_0^{\infty} w(x,t)H_{ij}(x,t) = \mu_{ij} ---------- (15)$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_0^{\infty} \varphi_2(m,r)\psi_{ij}(m,r)dmdr$$

$$H_{ij}(x,t) = \int_{a_1}^{b_1}\int_0^{\infty} K(m,r,x,t)\psi_{ij}(m,r)dmdr$$

We apply the truncated expansion method and find a numerical approximation $p_{2n}(x,t)$ for $w(x,t)$.

APPROXIMATION OF $p(x)$

We consider

$$w_{tt}(x,t) = w_{xx}(x,t) + f(t)p(x) ---------- (16)$$

We take as an auxiliary function

$$u(m,r,x,t) = e^{-m\left(\frac{x}{b_1}\right)} e^{-r(t)}$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w_t)$$

$$div(F^*) = w_{xx}(x,t) - w_{tt}(x,t) = -f(t)p(x)$$

Again we considerer

$$\iint_E u\,(-f(t)p(x))dA = \iint_E div(uF^*)dA - \iint_E F^*.\nabla u dA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_E F^*.\nabla u dA = \iint_E (F_1 u_x + F_2 u_t)dA$$

Integrating by parts:

$$\iint_E F_1\, u_x dA = \int_0^{\infty} \int_{a_1}^{b_1} F_1\, u_x dxdt =$$

$$= \int_0^{\infty} \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt - \iint_E w u_{xx}\, dA$$

Analogously

$$\iint_E F_2\, u_t dA = \int_{a_2}^{b_2} \int_{a_1}^{b_1} F_2\, u_t dxdt =$$

$$= \int_{a_1}^{b_1} \big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx - \iint_E w u_{tt}\, dA$$

then

$$\iint_D F^*.\nabla u dA =$$

$$= \int_{a_2}^{b_2} \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt +$$

$$+ \int_{a_1}^{b_1} \big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx -$$

$$-\iint_E w(u_{xx}+u_{tt})dA$$

where

$$\iint_E w(u_{xx}+u_{tt})dA=\iint_E wu\left(-\left(\frac{m}{b_1}\right)^2+(r)^2\right)dA$$

On the other hand,

$$\iint_E div(uF^*)dA=\int_C (uF^*).nds=$$

$$=\int_{a_1}^{b_1} u(m,r,x,b_2)w_t(x,a_2)dx-\int_{a_1}^{b_1} u(m,r,x,a_2)w_t(x,b_2)dx+$$

$$+\int_{a_2}^{b_2} u(m,r,b_1,t)w_x(b_1,t)dt-\int_{a_2}^{b_2} u(m,r,a_1,t)w_x(a_1,t)dt=G(m,r)$$

$$\therefore \iint u(-f(t)p(x))dA=G(m,r)-$$

$$-\int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,r,b_1,t)-w(a_1,t)u_x(m,r,a_1,t)\big)dt-$$

$$-\int_{a_1}^{b_1}\big(w(x,b_2)u_t(m,r,x,b_2)-w(x,a_2)u_t(m,r,x,a_2)\big)dx+\iint_E wu\left(-\left(\frac{m}{b_1}\right)^2+(r)^2\right)dA$$

We make $-\left(\frac{m}{b_1}\right)^2+r^2=0$, then (if $m\geq 0$, $r\geq 0$, $b_1>0$)

$$\iint u\left(m,\frac{m}{b_1},x,t\right)(-f(t)p(x))dA=G\left(m,\frac{m}{b_1}\right)-$$

$$-\int_{a_2}^{b_2}\left(w(b_1,t)u_x\left(m,\frac{m}{b_1},b_1,t\right)-w(a_1,t)u_x\left(m,\frac{m}{b_1},a_1,t\right)\right)dt-$$

$$-\int_{a_1}^{b_1}\left(w(x,b_2)u_t\left(m,\frac{m}{b_1},x,b_2\right)-w(x,a_2)u_t\left(m,\frac{m}{b_1},x,a_2\right)\right)dx$$
$$=\varphi_3(m)--------(17)$$

$$\iint u\left(m,\frac{m}{b_1},x,t\right)(-f(t)p(x))dA=\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\ p(x)\left(\int_0^{\infty} e^{-\frac{m}{b_1}t}\ f(t)dt\right)dx=\varphi_3(m)$$

If $K(m)=\int_0^{\infty} e^{-\frac{m}{b_1}t}\ f(t)dt$ then

$$\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\ p(x)dx=\frac{\varphi_3(m)}{K(m)}\ ---------(18)$$

We take a base:

$$\psi_i(m)=m^i e^{-m}\qquad i=0,1,\dots,n$$

and we multiply both members of (18) by $\psi_i(m)$ and we integrate with respect to m.

We have then the generalized moments problem

$$\int_{a_1}^{b_1} p(x)H_i(x)=\mu_i\ ----------\ (19)$$

where

$$\mu_i=\int_{a_1}^{b_1}\frac{\varphi_3(m)}{K(m)}\ \psi_i(m)dm$$

$$H_i(x)=\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\ \psi_i(m)dm$$

We apply the truncated expansion method and find a numerical approximation $p_{3n}(x)$ for $p(x)$.

If source is $p(x)f(x,t)$ then

$$\iint u\left(m,\frac{m}{b_1},x,t\right)(-f(x,t)p(x))dA=\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\ p(x)\left(\int_0^{\infty} e^{-\frac{m}{b_1}t}\ f(x,t)dt\right)dx=\varphi_3(m)$$

We wrote $K(m,x)=\int_0^{\infty} e^{-\frac{m}{b_1}t}\ f(x,t)dt$ then

$$\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\ H(m,x)\ p(x)dx=\varphi_3(m)\ ---------(20)$$

If $\left\{e^{-\frac{m}{b_1}x} H(m,x)\right\}_m$ are linearly independent then we apply the truncated expansion method and find a numerical approximation $p_{3n}(x)$ for $p(x)$.
For this it must happen that $f(x,t)$ be such that $\left|\min_{i,x} H(i,x)\right| \neq 0$ given that

$$0 = \left|\sum_{m=1}^{n} k_m e^{-\frac{m}{b_1}x} H(m,x)\right| \geq \left|\sum_{m=1}^{n} k_m e^{-\frac{m}{b_1}x}\right| \left|\min_{i,x} H(i,x)\right|$$

$$\therefore \sum_{m=1}^{n} k_m e^{-\frac{m}{b_1}x} = 0 \quad \Longrightarrow k_m = 0 \qquad m = 1,\ldots,n$$

NUMERICAL EXAMPLES

Example 1

We consider the equation

$$w_{tt}(x,t) = w_{xx}(x,t) + p(x)(-e^{-t}) \qquad E = (0,1) \times (0,\infty)$$

The solution is : $w(x,t) = e^{-(1+x)^2-t}$ $\qquad p(x) = e^{-(1+x)^2}(1 + 8x + 4x^2)$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$, and
$p(x)$ with $p_{3n}(x)$

To find $p_{1n}(x,t)$ we take $n = 6$ moments $(m = 0, \ldots, 5)$
To find $p_{2n}(x,t)$ we take $n = 6$, moments $n_1 = 3, n_2 = 2$, in total $n = n_1.n_2 = 6$ moments.

$$\sqrt{\int_0^\infty \int_0^1 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.0243825$$

In the Figure 1 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x)$ we take $n = 5$ moments

$$\sqrt{\int_0^1 \left(p(x) - p_{3n}(x)\right)^2 dx} = 0.00110864$$

In the Figure 2 we show the exact solution (light blue) and the approximate solution (magenta color).

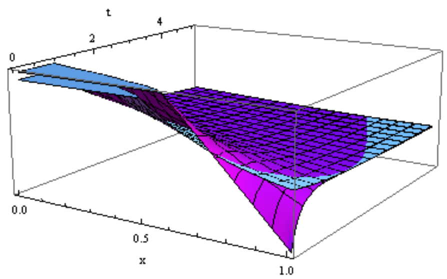

Fig. 1: $w(x,t)$ and $p_{2n}(x,t)$

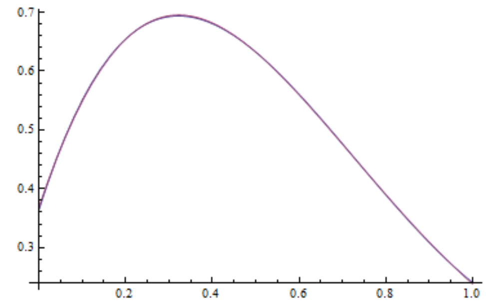

Fig. 2: $p(x)$ and $p_{3n}(x,t)$

Example 2

We consider the equation

$$w_{tt}(x,t) = w_{xx}(x,t) + p(x)e^{-t-x} \qquad E = (0,1) \times (0,\infty)$$

The solution is : $w(x,t) = (x+1)^2(e^{-t-x})$ $\qquad p(x) = (2+4x)$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$, and
$p(x)$ with $p_{3n}(x)$

To find $p_{1n}(x,t)$ we take $n = 6$ moments $(m = 0, \ldots, 5)$
To find $p_{2n}(x,t)$ we take $n = 6$, moments $n_1 = 3, n_2 = 2$, in total $n = n_1.n_2 = 6$ moments.

$$\sqrt{\int_0^\infty \int_0^1 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.105668$$

In the Figure 3 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x)$ we take $n = 5$ moments

$$\sqrt{\int_0^1 \left(p(x) - p_{3n}(x)\right)^2 dx} = 0.004861$$

In the Figure 4 we show the exact solution (light blue) and the approximate solution (magenta color).

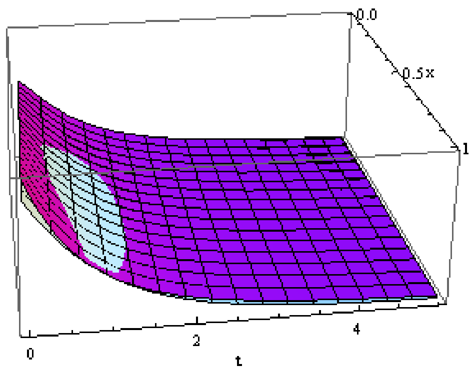

Fig. 3: $w(x,t)$ and $p_{2n}(x,t)$

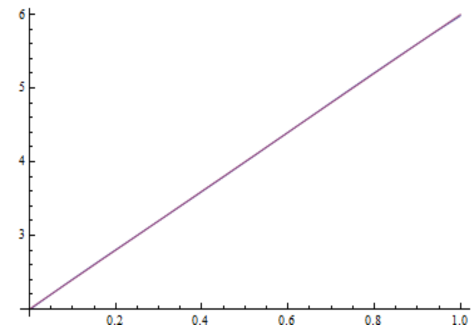

Fig. 4: $p(x)$ and $p_{3n}(x,t)$

CONCLUSIONS

The problem of finding $w(x,t)$ and $p(x)$ in the Poisson equation $w_{tt} = w_{xx} + f(t)p(x)$ about a domain $E = (a_1, b_1) \times (a_2, \infty)$, under the Cauchy conditions it is possible to solve it using the problem generalized moments techniques. First we can deduce $w(x,t)$ in two steps. We consider the equation $w_{tt} = w_{xx} + f(t)p(x)$, we turn it into an integral equation and with the techniques of inverse problem moments we find an approximate solution $p_{1n}(x,t)$ for $-w_x + w_t$. Then we consider the equation $-w_x(x,t) + w_t(x,t) = p_{1n}(x,t)$ and again we take it to an integral equation to solve it and find an approximate solution $p_{2n}(x,t)$ for $w(x,t)$. Finally we consider $w_{tt} = w_{xx} + f(t)p(x)$ and again we take it to the integral equation $\int_{a_1}^{b_1} e^{-\frac{m}{b_1}x}\, p(x)dx = \frac{\varphi_3(m)}{K(m)}$ and find an approximate solution $p_{3n}(x)$ for $p(x)$.

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