POPULATION GROWTH USING PSEUDO-LOGISTIC MODEL

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Abstract

Malthusian, Verhulst and Richards Growth Models are widely used population growth models. However, in this article, we simulate the future population growth of Federal Capital Territory (F.C.T) using three models: exponential, logistic and Richards; to ascertain the model that has the best fit in modelling population growth of FCT over time. This study is based on the assumption that population growth follows a sigmoid curve that shows the rate of population growth slowing down as the population approaches a certain limit or carrying capacity. Then, a comparative analysis of exponential, logistic and Richards methods is presented. Furthermore, the result of the analysis shows that FCT has a growing population and that Richards growth model with MAPE and RMSE values of 1.39% and 19159.53826 respectively is the most accurate and closely followed by logistic growth model with values 3.18% and 63451.47975 respectively. The study concludes that Richards growth model with R squared value of 0.786 has the best fit for population growth projection of FCT. With approximate growth rate at 9.3% per annum, the projected population of the FCT will hit 31,635,270 million by the year 2045 all things being equal. Therefore, the government should invest in massive agricultural revolution to accommodate the growing population.

Keywords: *carrying capacity, population size, population density, census, vital statistics.*

1. INTRODUCTION

Knowing the population of a state is a key factor used by policy makers to plan for the equi-table distribution of resources and strategically develop a blueprint for socio-economic devel-opment of the people. However, post-independence Nigeria conducted census exercises across the nation in 1962, 1973, 1991 and 2006 for the purpose of gathering relevant information such as growth rate, carrying capacity and other vital statistics. The National Population Commis- sion in conjunction with National Bureau of Statistics [7, 8, 6] published a population growth projection for 2007 to 2016 but failed to categorically state the approach that was used to arrive at the estimations. Presently there are no current population data for the F.C.T apart from the 1991–2016 years obtained from the National Population Commission; which necessitated this study to obtain a result that will fill up the gap. In an attempt to fill the gap, [5] used a mathemat- ical approach and compared exponential, geometric and hyperbolic population growth models to ascertain the model with the best fit and concluded that the exponential growth model per-formed better than other models in estimating the population growth projection. In 2023, [4]compared exponential growth and logistic growth models to ascertain the model with the best fit and concluded that logistic model performed better. Conversely, in this article, we compared the exponential, logistic and Richards growth models to ascertain the model with the best fit. To achieve this, we employed the Mean Absolute Percentage Error (MAPE), Root Mean Square Error (RMSE) and R squared values in the analysis. Meanwhile, population growth refers to the increase in population size resulting from variations in birth and death rates, measured both in absolute and relative terms. On the other hand, the population growth rate (PGR) represents the average annual percentage change in population size [1], influenced by factors such as birth, death, immigration, and emigration within a specific period. In 2003, [2] argued that availability of food is a major factor which affect growth rate of a people and put the Malthusian parameter at $\alpha = 0.03$. Present-day challenges such as high unemployment rates, rising poverty, and crime levels, increased cost of living, and environmental degradation are indicators that the F.C.T is being overpopulated.

1.1. **Study Area.** Federal Capital Territory (FCT), Abuja is the capital of Nigeria and geographically located at the center, lying between latitudes $8^0 25'$ and $9^0 20'$ north of the equator and longitudes $6^0 45'$ and $7^0 39'$ east of Greenwich Meridian covering a landmass of approx- imately 7, 315km^2, with population density of 190/km^2. F.C.T is one of the fastest growing cities in the world with a projected population of over 3 million with an annual growth rate of 10% in 2016 [6]. The Federal Capital territory is located north of the confluence of River Niger and River Benue. It is bordered by the states of Niger to the West and North, Kaduna to the northeast, Nasarawa to the east and south, and Kogi to the southwest. The indigenous tribe in FCT includes among others; Koro, Gbari, Gade, Nupe, Gwandara, Dibo, Bassa , Ganagana, and Ebira, and also settler groups like the Hausa, Igbo and Yoruba[9]. The Federal Capital Territory was created out from Niger, Plateau and Kwara states upon the promulgation of decree number 6 of 1976 by General Murtala Mohammed. The Federal capital territory is currently made up of six local government areas, namely; Abaji, Abuja Municipal, Bwari, Gwagwalada, Kuje and Kwali. The Federal Capital Territory Administration is headed by a minister, who is appointed by the president.

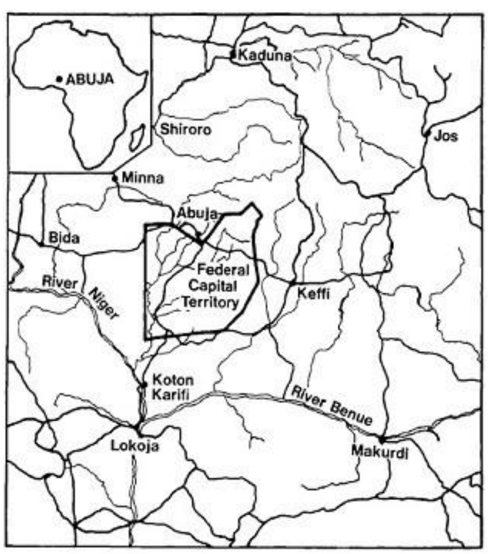

FIGURE 1. Map of the Federal Capital Territory (FCT)

2.MATERIAL AND METHODS

Exponential Growth Model. The first mathematical model describing population growth is the Malthus model or exponential model [3]. Let N be the population size at time t. In the exponential model, the growth rate r is assumed proportional to the size of existing population $N(t)$. The exponential model could be represented by the following differential equation

$$\frac{dN}{dt} = rN$$

with initial condition $N(0) = N_0$. Integrating and multiplying both sides by $\frac{dt}{N}$ gives

$$\int_0^t \frac{1}{N} = \int_0^t rsds$$

Therefore,

$N(t) = N_0 e^{rt}$ (2.4)

where the number of population at time t is $N(t)$, $N(0) = N_0$ is the initial population size, r isthe population growth rate and $t = time$. However, the exponential growth model is rarely used to describe population growth, since it produces an unbounded population growth and the

$$\lim_{t \to \infty} N_0 e^{rt} = \infty.$$

Logistic Growth Model. The exponential growth model was improved upon by logistic growth model. In the logistic model, a population grows until it attains a maximum capacity [1]. The logistic model is based on the assumption that the growth rate *dN/dt* is proportional to the existing population and the remaining resources available to the existing population. Verhulst [4] was the first to express logistic growth model as:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \quad (2.5)$$

where r is the population growth rate and K is the carrying capacity for $N(t)$ as t ∞. The solution to the logistic equation can be obtained by applying the separation of variables method. From Eq. (2.5), we have:

$$N(t) = \frac{K}{1 + Ae^{-rt}}$$

Where

$$A = \frac{K - N_0}{N_0}$$

Then, the equation above is the classical logistic growth model. Furthermore, the logistic model can be modified and written as

$$N(t) = \frac{K}{1 + Ak^t}$$

Where $k = e^{-r}$.

The logistic model predicts rapid initial growth and then a decrease in growth rate as time goes on so that the size of the population approaches a limit. The above Eq. (2.19) can be modified and written as

$$N(t) = \frac{K}{1+Ae^{-rt}}. \tag{2.21}$$

Replace the exponent with $1/1-m$ where $m \neq 1$ is a parameter that in part determines the point of inflection on the $y-$axis, then Eq. (2.21) will be

$$N(t) = \left(\frac{K}{1+Ae^{-rt}}\right)^{1/1-m}. \tag{2.22}$$

Hence, Eq. (2.22) is known as Richards Growth model.

General assumptions for the models.

(a) All data described in this work are real data.
(b) Availability of food is proposed as the variable which best accounts for the human carrying capacity
(c) All individuals are identical
(d) No impact of other species on population growth
(e) No time lag in effect of density on population growth.

Uncertainty Quantification. Uncertainty in modeling arises from errors, assumptions, and approximations made during the experimental process. This uncertainty can be categorized into two types:

(a) Mean Absolute Percentage Error (MAPE): this is a relative error measure that uses relative errors to enable us to compare forecast accuracy between time-series models.
(b) Root Mean Square Error (RMSE): this is the standard deviation of the prediction errors. RMSE is a measure of how spread out these prediction errors are. In other words, it tells you how concentrated the data is around the line of best fit.

The Mean Absolute Percentage Error (MAPE), Root Mean Square Error (RMSE) and *R*-squared value for the exponential, logistic and Richards models were computed to obtain the most ac- curate model by using the following formula;

$$\mathrm{MAPE} = \frac{1}{\mathrm{N}}\sum_{\mathrm{t}=1}^{\mathrm{N}}\left|\frac{\mathrm{A_t}-\mathrm{F_t}}{\mathrm{A_t}}\right| \times 100\%$$

where A_t denotes actual value, F_t denotes forecast value and N is the number of observations. The Root Mean Square Error calculation is given by the formulae

$$\mathrm{RMSE} = \sqrt{\sum_{\mathrm{t}=1}^{\mathrm{N}}\frac{(\mathrm{A_t}-\mathrm{F_t})^2}{\mathrm{N}}}$$

A_t denotes actual value, F_t denotes forecast value and N is the number of observations. According to [5] *MAPE* < 10% is highly accurate, 10%-20% is good and 21%-50% is reasonable forecasting, while > 50% is inaccurate forecasting. However, the lower MAPE values are better because they signify smaller percentage errors.

R–**Squared.** *R* squared is a statistical metric that evaluates the extent to which the data align with the regression model. It measures the goodness of fit, indicating the accuracy of predictions made using the model. A higher $R-$squared value

TABLE 1. Model parameters and their interpretation

S/N	Parameter	Symbols
1	Initial population	N_0
2	The population at t	$N(t)$
3	Growth rate	r
4	Carrying capacity	K
5	Actual value	A_t
6	Forecast value	F_t
7	Number of observation	N
8	The sum of squares due to regression	$SS_{regression}$
9	The total sum of squares	SS_{total}

indicates a better fit of the data, with values closer to 1 indicating a stronger fit. The significance of $R-$squared lies in its capacity to estimate the probability of future events falling within the predicted outcomes. The formula for calculating $R-$squared is as follows:

$$\mathrm{R-squared} = 1 - \frac{\sum(y_i - \hat{y})^2}{\sum(y_i - \bar{y})^2}$$

where:

(a) $SS_{regression}$ is the sum of squares due to regression: the sum of squares due to regres- sion quantifies the degree to which the regression model accurately represents the data utilized for modeling

(b) SS_{total} is the total sum of squares: the total sum of squares measures the overall variation present in the observed data used for regression modeling (see Table 2)

3.RESULTS AND DISCUSSION

In this section, we engage in parameter estimation, simulation, and the subsequent discussion of the results of the exponential, logistic and Richards models, along with their corresponding uncertainty quantifications. The data obtained serves as the basis for these analyses. Through parameter estimation, we determine the values of the model parameters, enabling us to simulate and generate predictions. Subsequently, we evaluate the outcomes of these models and consider the associated uncertainties. Using Matlab $R2023a$, we run the data codes, plot the results, estimate parameters, fit the model and the data, carry out the error analysis for the population growth projection of F.C.T annual population for the years (1991–2016), calculate the goodness of fit for all the models using $R-$squared, and then simulate.

Population data visualization. The population data visualization for F.C.T (Abuja) from 1962–2019 is plotted as demonstrated in Fig. 2 using Matlab $R2023a$;

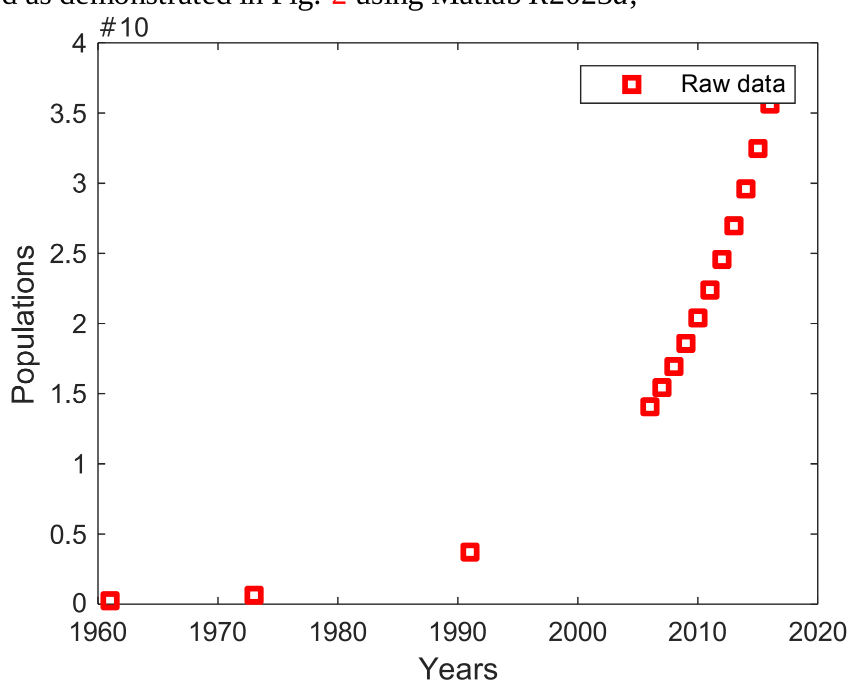

FIGURE 2. Population raw data for F.C.T

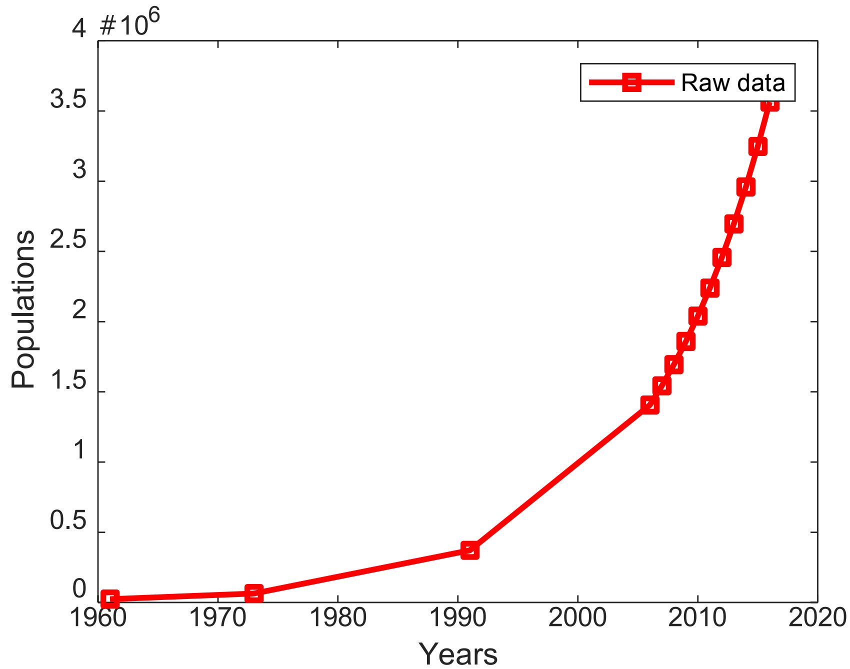

FIGURE 3. Population raw data simulation for F.C.T

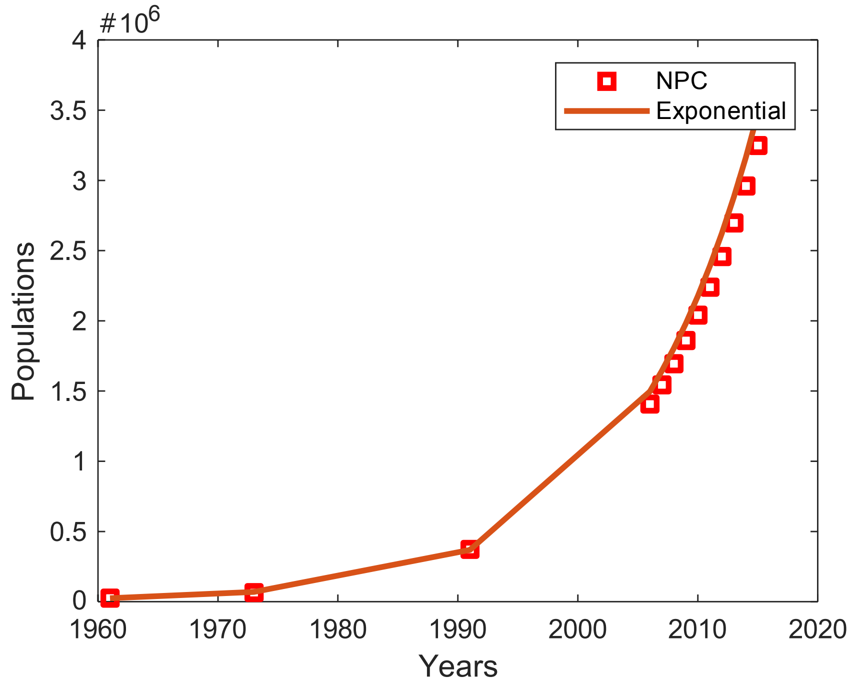

FIGURE 4. Exponential Model simulation for F.C.T

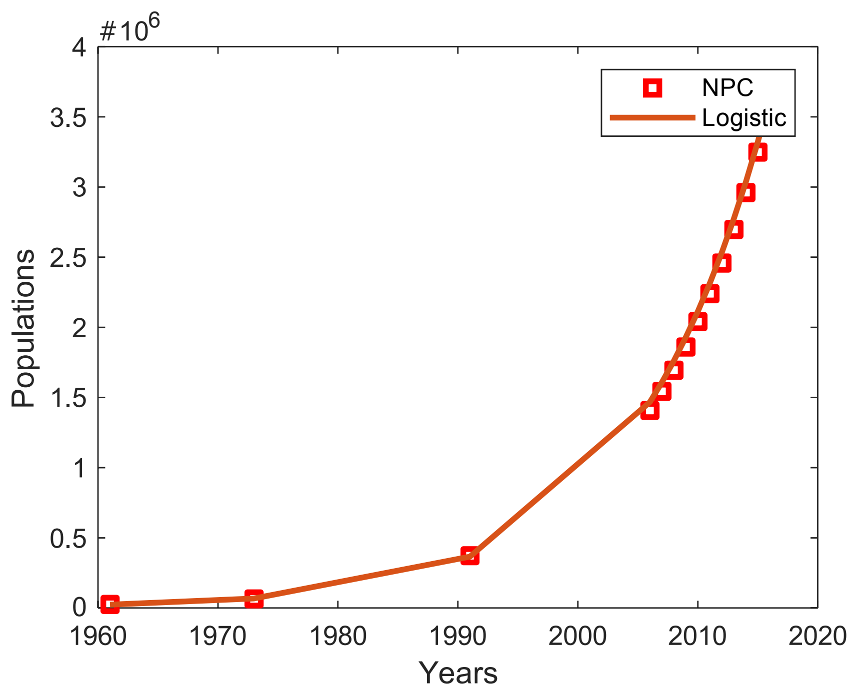

FIGURE 5. Logistic model simulation for F.C.T

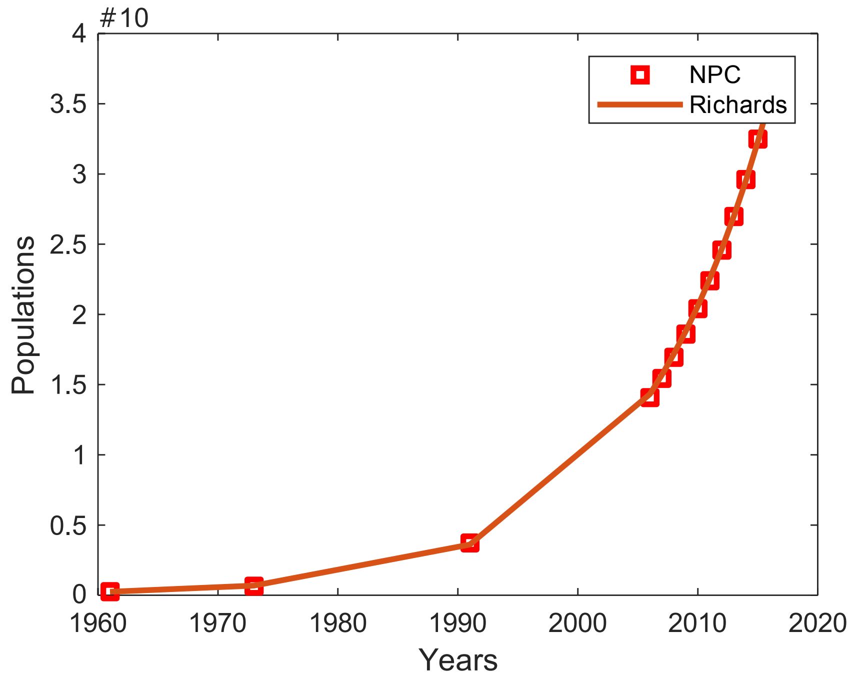

FIGURE 6. Richards Model simulation for F.C.T

Estimation of parameters. In this section, we estimated the parameters involved for the models and obtained the following values below. All codes were written in Matlab $R2023a$, and installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 GB (7.78 GB usable) RAM running on Windows 10 operating system.

Example 3.1. From Table 3 the population of F.C.T in 1962 and 1973 is 24,314 and 67,677 people respectively. Then, considering that the population of F.C.T in 1991 and 2006 was actually 377,674 and 1,406,239 respectively. Hence, by applying Eq. (2.19), we have

$$N(11) = \frac{K}{1+(K-24{,}314)e^{11r}/24{,}314} = 67{,}677 \tag{3.1}$$

$$N(29) = \frac{K}{1+(K-24{,}314)e^{29r}/24{,}314} = 377{,}674 \tag{3.2}$$

we can obtain the carrying capacity K and population growth rate r from solving these systemof equations (3.1) and (3.2) simultaneously. Hence, let

$$K = 74{,}483{,}314, \qquad r = 0.093063097449163, \qquad A = 3062.39203750$$

3.2.1. *Error Analysis.* The table 2 below shows the results of the error analysis conducted on exponential, logistic and Richards population growth models using the simulated data of F.C.T from 1962 to 2016.

TABLE 2. Error Analysis of different Growth Models

Error Analysis	Exponential	Logistic	Richards
%MAPE	5.842163891	3.1845749361	1.395131277
RMSE	147046.7467	63451.47975	19159.53826
$R-$Squared	0.773	0.784	0.786
Linear regression	$y = 64279x - 66542$	$y = 61761x - 62943$	$y = 60037x - 60938$

Simulations. We present the F.C.T population size simulation from 2006–2045 in this subsection using the exponential, logistic and Richards models.

TABLE 3. Population simulation using different Growth Models

Year	Time (t)	N.P.C Data	Exponential	Logistic	Richards
2006	44	1,406,239	1,495,758	1,466,781	1,431,464
2007	45	1,543,293	1,642,557	1,607,629	1,568,122
2008	46	1,693,706	1,803,764	1,762,675	1,717,519
2009	47	1,858,777	1,980,792	1,930,093	1,880,783
2010	48	2,039,937	2,175,194	2,114,143	2,059,127
2011	49	2,238,752	2,388,676	2,315,184	2,253,858
2012	50	2,456,945	2,623,109	2,534,672	2,466,377
2013	51	2,696,403	2,880,551	2,774,169	2,698,186
2014	52	2,959,199	3,163,259	3,035,341	2,950,889
2015	53	3,247,608	3,473,713	3,319,963	3,226,197
2016	54	3,564,126	3,814,637	3,629,917	3,525,923
2020	58		5,547,406	5,164,452	5,008,987
2025	63		8,859,026	7,919,650	7,670,891
2030	68		14,147,574	11,892,551	11,512,976
2035	73		22,593,211	17,339,298	16,795,671
2040	78		36,080,618	24,311,681	23,594,095
2045	83		57,619,563	32,493,509	31,635,270

Exponential model simulation. We present the simulation of the population size of F.C.T (Abuja) from 1962 to 2045 as demonstrated in Table 3, whose graph is shown in Fig. 4. The orange colored line represents the population simulation of F.C.T using the exponential model; this figure shows that the initial population of F.C.T is 24,314 people in the year 1962, while by the year 2045, the population of F.C.T will be 57,619,563 million people. The results from the exponential model shows that the population growth rate of F.C.T is 9.3% per annum. The exponential model predicts a growing population for the F.C.T with the carrying capacity of $K = 74,\ 483,\ 314$ if other factors are constant. We calculated the goodness of fit for the exponential model using $R-$squared and obtained the $R-$ squared value of 0.773. Since the calculated R-squared value for the model is positive and approaches 1, it implies that the model is significant and the fitting is very accurate. Using the exponential growth model and the values of its parameters, the Mean Absolute Percentage Error (MAPE) is approximately 5.8% and Root Mean Square Error (RMSE) is 147046.7467.

Logistic model simulation. We present the simulation of the population size of F.C.T from 1962 to 2045 as demonstrated in Table 3, whose graph is shown in Fig. 5. The orange colored line represents the population simulation of F.C.T using the logistic model; this figure shows that the initial population of F.C.T is 24,314 people in the year 1962, while by the year 2045, the population of F.C.T will be 32,493,509 million people. The results from the logistic model show that the population growth rates of F.C.T is 9.3% per annum. The Logistic model predicts a growing population for the F.C.T with the carrying capacity of K = 74,483,314 if other factors are constant. We calculated the goodness of fit for the logistic model using R squared and obtained the R-squared value of 0.784. Since the calculated R squared value for the model is positive and approaches 1, it implies that the model is significant and the fitting is very accurate. Using the logistic model and the values of its parameters, the Mean Abso- lute Percentage Error (MAPE) is approximately 3.2% and Root Mean Square Error (RMSE) is 63451.47975.

Richards model simulation. We present the simulation of the population size of F.C.T from 1962 to 2045 as demonstrated in Table 3, whose graph is shown in Fig. 6. The orange colored line represents the population simulation of F.C.T using the Richards model; this figure shows that the initial population of F.C.T (Abuja) is 24,317 people in the year 1962, while by the year 2045, the population of F.C.T will be 31,635,270 million people. The results from the Richards model show that the population growth rates of F.C.T is 9.3% per annum. The Richards model predicts a growing population for the F.C.T with the carrying capacity of K = 74,483,314 if other factors are constant. We calculated the goodness of fit for the logistic model using R squared and obtained the R squared value of 0.786. Since the calculated R squared value for the model is positive and approaches 1, it implies that the model is significant and the fitting is very accurate. Using the Richards model and the values of its parameters, the Mean Absolute Percentage Error (MAPE) is approximately 1.4% and Root Mean Square Error (RMSE) is 19159.53826.

4. CONCLUSION

The exponential, logistic and Richards models predict a growing population for the F.C.T with the carrying capacity of K = 74,483,314 and population growth rate at 9.3%, if other factors are constant. Suppose the population growth moves closer and closer to the carrying capacity, and there are no developmental plans and policies to engage and accommodate the population growth. In that case, there will be a crowding effect on the population density, which will bring about competition, resulting in hardship and hunger. Eventually, this will give birth to crises, insecurity, kidnapping, and other societal crimes. Regarding these factors mentioned, it is expedient that the government give attention to the population growth of the

F.C.T by implementing plans and policies geared towards industrialization and engaging the population in meaningful developmental projects such as agriculture. A growing population when strategically harnessed can build a strong economy and the manpower that can build a strong army to defend its territory. From the results of the comparative analysis obtained from the three models, all three models are significant, and their fitting is very accurate, but the Richards model has a better fitting and more accurate because of its R squared value of 0.786 which is very close to 1 as compared to the logistic model R-squared value of 0.784 and exponential model $R-$squared of 0.773. The uncertainty about the parameters within the

Richards model has captured fewer errors and has quantified better confidence in predictions with mean absolute percentage error of 1.4%, compared to its counterparts, the exponential model with 5.8% and logistic model with 3.2%. We can therefore make better decisions with known levels of confidence that both logistic and Richards models predicted results are more precise and very accurate for the future population growth projection of the F.C.T.

Conflicts of interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported inthis paper.

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