ON SOME NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SALAGEAN OPERATOR

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Abstract

In this work, two new subclasses of bi-univalent functions $M_{\Sigma}^{n}(\alpha,\lambda)$ and $M_{\Sigma}^{n}(\lambda,\beta)$ using the Salagean differential operator are discussed. The coefficients estimates for $|a_2|$ and $|a_3|$ and the upper bounds for the Fekete-Szego functionals for functions belonging to the classes are obtained. Results obtained generalized some known results.

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INTRODUCTION

Let A denotes the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C: |z| < 1\}$. Let S be the subclass of A consisting of functions which are analytic and univalent in $\mathbb{U}$.

Some of the important and well known subclasses of univalent function class S include (for example) the class $S^*(\beta)$ of starlike functions of order β in $\mathbb{U}$ and the class $K(\beta)$ of convex function of order β in $\mathbb{U}$. By definition we have

$$S^*(\beta) := \left\{f: f \in S \quad and \quad Re\left(\frac{zf\prime(z)}{f(z)}\right) > \beta; z \in \mathbb{U}; 0 \leq \beta < 1\right\} \tag{2}$$

and

$$K(\beta) := \left\{ f : f \in S \quad and \quad Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta; z \in \mathbb{U}; 0 \leq \beta < 1 \right\}. \tag{3}$$

It readily follows from the definition (2) and (3) that $f \in K(\beta) \Leftrightarrow zf'(z) \in S^*(\beta)$.

Its is well known by Keobe One-Quarter Theorem [4] that the range of every function of the class S contains the disk $\{w: |w| < 1/4\}$. Therefore every $f \in S$ has an inverse function f^{-1} satisfying $f^{-1}(f(z)) = z \; z \in \mathbb{U}$ and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f)); r_0(f) \geq 1/4$$

The inverse of $f(z)$ has a series expansion in some disc about the origin of the form

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots \tag{4}$$

A function $f(z)$ univalent in a neighbourhood of the origin and its inverse satisfy the condition $f(f^{-1}(w)) = w$ using (4) yields

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots, \tag{5}$$

now using (5) we get the following results

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{6}$$

An analytic function $f(z)$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. The class of analytic bi-univalent function in $\mathbb{U}$ is denoted by Σ.

Example of functions in the class Σ are

$$\frac{z}{1-z}; \quad -log(1-z); \quad \frac{1}{2} log\left(\frac{1+z}{1-z}\right).$$

However the familiar Keobe function $\frac{z}{(1-z)^2}$ is not a member of the class Σ. (see[16])

Other example of function that does not belong to Σ are

$$z - \frac{z^2}{2} \; ; \; \frac{z}{1-z^2}$$

(see[16]).

Lewin [8] investigated the bi-univalent class of functions in Σ and showed that from (1) the $|a_2| < 1.51$. Netanyahu [13] on the other hand showed that from (1) the $\text{Max}|a_2| < 4/3$. Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Brannan and Taha [3] introduced these subclasses $S_\Sigma^{*\beta}, S_\Sigma^*(\rho), K_\Sigma(\alpha), K_\Sigma(\beta)$ of bi-univalent class of functions Σ Similar to the two subclasses $S^*(\alpha)$ and $K(\alpha)$ of the univalent functions in class S (see[2]) which are define as follows. Thus Brannan and Taha [3] (see also [17]), a function $f \in A$ is in the class $S_\Sigma^{*\beta}$, $(0 < \beta \leq 1)$, the class of strongly bi-starlike functions of order β if each of the following conditions are satisfied

$$f \in \Sigma, \qquad \left|arg\left(\frac{zf\prime(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left|arg\left(\frac{wg\prime(w)}{g(w)}\right)\right| < \frac{\beta\pi}{2} \qquad (w \in \mathbb{U})$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

Similarly, a function $f \in A$ is in the class $K_\Sigma(\alpha), (0 < \alpha \leq 1)$ the class of strongly bi-convex function of order α if each of the following conditions are satisfied

$$f \in \Sigma, \qquad \left|arg\left(1 + \frac{zf\prime\prime(z)}{f\prime(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left|arg\left(1 + \frac{wg\prime\prime(w)}{g\prime(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U})$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

The classes $S_\Sigma^*(\rho)$ and $K_\Sigma(\beta)$ of bi-starlike functions of order ρ and bi-convex functions of order

β corresponding (respectively) to the function classes $S^*(\alpha)$ and $K(\beta)$ defined by (2) and (3), were also were also analogously. For each function $S_{\Sigma}^{*\beta}$ and $K_{\Sigma}(\alpha)$, Brannan and Taha [3] found non sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for detail see [3,17]). Li and Wang [9] introduced and studied two subclasses called the λ-convex functions for Σ and find the estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. Recently, many author investigated bounds for various subclasses of bi-univalent functions ([7,12,11,10,6,14]). But the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in N\{1,2\}; N = \{1,2,3 \cdots\}$ is presumably still an open problem. In 1983 Salagean [15] introduced a differential operator $D^n: A \to A$ defined by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

$$= z(D^{n-1} f(z))', \qquad n \in \mathbb{N}_0 \cup \{0\},$$

note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad n \in \mathbb{N}_0 = \mathbb{N}_0 \cup \{0\}.$$

The purpose of this work, is to investigate bi-univalent properties of two subclasses of functions defined by a Salagean differential operator. Coefficients estimates for $|a_2|$ and $|a_3|$ and Fekete-Szego functional estimates using the techniques of Srivatava et al. [16] and Zaprawa [18] respectively are established.

Let P be the class of Caratheodory functions i.e P is the family of functions φ analytic in U for which

$Re\{\varphi(z)\} > 0$ $\varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in U$.

Lemma 1.1 *([4]). If $\varphi \in P$ then*

$$\varphi(z) = 1 + \Sigma_{k=1}^{\infty} c_k z^k$$

and $|c_k| \leq 2$ $k = 1,2 \cdots$. This inequality is sharp for each k.

COEFFICIENT BOUNDS AND FEKETE-SZEGO FUNCTIONAL ESTIMATE FOR THE CLASS $M_{\Sigma}^{n}(\alpha, \lambda)$

Definition 2.1 A function $f(z)$ given by (1) is said to be in $M_\Sigma^n(\alpha,\lambda)$ if the following conditions are satisfied

$$f \in \Sigma,$$

$$\left|arg\left\{(1-\lambda)\frac{D^{n+1}f(z)}{D^n f(z)}+\lambda\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\}\right| < \frac{\alpha\pi}{2} \tag{7}$$

$$(0<\alpha\le 1, \lambda \ge 0, n \in N_0 : z \in U)$$

and

$$\left|arg\left\{(1-\lambda)\frac{D^{n+1}g(w)}{D^n g(w)}+\lambda\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right\}\right| < \frac{\alpha\pi}{2} \tag{8}$$

$$(0<\alpha\le 1, \lambda \ge 0, n \in N_0 : w \in U), N_0 = N \cup \{0\}$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{9}$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $M_\Sigma^n(\alpha,\lambda)$.

Theorem 2.1 *Let $f(z)$ given by (1) be in the class $M_\Sigma^n(\alpha,\lambda)$, then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1+2\lambda)3^n - [2\alpha(1+3\lambda)+(\alpha-1)(1+\lambda)^2]2^{2n}}} \tag{10}$$

$$|a_3| \le \frac{\alpha}{(1+2\lambda)3^n} + \frac{4\alpha^2}{(1+\lambda)^2 2^{2n}} \tag{11}$$

$(0<\alpha\le 1, n \in N_0 = N\cup\{0\})$.

Proof. The inequalities from (7) and (8) are equivalent to

$$(1-\lambda)\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)+\lambda\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right) = [p(z)]^\alpha \tag{12}$$

and

$$(1-\lambda)\left(\frac{D^{n+1}g(w)}{D^{n}g(w)}\right)+\lambda\left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)=[q(w)]^{\alpha} \tag{13}$$

where $p(z)$ and $q(w)$ are in the class P and have the forms

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{14}$$

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots \tag{15}$$

Now equating the coefficients in (12) and (13) yields

$$(1+\lambda)2^n a_2=\alpha p_1, \tag{16}$$

$$2(1+2\lambda)a_3 3^n-(1+3\lambda)2^{2n}a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)p_1^2}{2!}, \tag{17}$$

$$-(1+\lambda)2^n a_2=\alpha q_1, \tag{18}$$

and

$$2(1+2\lambda)(2a_2^2-a_3)3^n-(1+3\lambda)2^{2n}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)q_1^2}{2!}. \tag{19}$$

From (16) and (18)

$$p_1=-q_1 \tag{20}$$

and

$$2(1+\lambda)^2a_2^2 2^{2n}=\alpha^2(p_1^2+q_1^2). \tag{21}$$

From (17), (19) and (21) the following are obtained

$$4(1+2\lambda)a_2^2 3^n-2(1+3\lambda)a_2^2 2^{2n}=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2!}\left(\frac{2(1+\lambda)^2a_2^2 2^{2n}}{2\alpha}\right). \tag{22}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{4\alpha(1+2\lambda)3^n-(2\alpha(1+3\lambda)+(\alpha-1)(1+\lambda)^2)2^{2n}}. \tag{23}$$

Applying Lemma (1.1) to the coefficients p_2 and q_2 in (23) yields

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1+2\lambda)3^n-(2\alpha(1+3\lambda)+(\alpha-1)(1+\lambda)^2)2^{2n}}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (19) from (17) yields

$$4(1+2\lambda)a_3 3^n - 4(1+2\lambda)a_2^2 3^n = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2!}(p_1^2-q_1^2). \tag{24}$$

Its follows from (20), (21) and (24) that

$$a_3 = \frac{\alpha(p_2-q_2)}{4(1+2\lambda)3^n} + \frac{\alpha^2(p_1^2+q_1^2)}{2(1+\lambda)^2 2^{2n}} \tag{25}$$

Applying Lemma (1.1) to the coefficients of p_1, p_2, q_1 and q_2 in (25) yields

$$|a_3| \leq \frac{\alpha}{(1+2\lambda)3^n} + \frac{4\alpha^2}{(1+\lambda)^2 2^{2n}}$$

Putting $n=0$ in theorem 2.1 we have

Corollary 2.1 *Let $f(z)$ given by (1) be in the class $M_\Sigma(\alpha,\lambda)$. Then.*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1-\lambda)(\alpha+\lambda+1-\alpha\lambda)}}; |a_3| \leq \frac{4\alpha^2}{(1-\lambda^2)} + \frac{\alpha}{1+2\lambda}$$

which are the results obtained by Li and Wang [9].

Putting $\lambda=0$ and $n=0$ in theorem 2.1 we have

Corollary 2.2 *Let $f(z)$ given by (1) be in the class $S_\Sigma^{*\alpha}$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}; \qquad |a_3| \leq 4\alpha^2 + \alpha.$$

Putting $\lambda = 1$ and $n = 0$ in theorem 2.1 we have

Corollary 2.3 *Let $f(z)$ given by (1) be in the class $K_\Sigma(\alpha)$. Then*

$$|a_2| \leq \alpha; \qquad |a_3| \leq \alpha^2 + \frac{\alpha}{3}.$$

In what follows, the Fekete Szego functional estimate for the class $M_\Sigma^n(\alpha,\lambda)$

Theorem 2.2 *Let $f(z)$ be of the form (1) be in $M_\Sigma^n(\alpha,\lambda)$,then $\mu \in \mathbb{R}$ $0 < \alpha \leq 1$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \alpha|h(\mu)| & \text{for}|\mathrm{h}(\mu)| \geq \frac{1}{(1+2\lambda)3^n} \\ \frac{\alpha}{(1+2\lambda)3^n} & \text{for} 0 \leq |\mathrm{h}(\mu)| \leq \frac{1}{(1+2\lambda)3^n} \end{cases} \tag{26}$$

where

$$h(\mu) = \frac{4\alpha(1-\mu)}{4\alpha(1+2\lambda)3^n - (2\alpha(1+3\lambda) + (\alpha-1)(1+2\lambda)^2)2^{2n}}.$$

Proof. From (16),(17),(18) and (19) the following were obtained

$$(1+\lambda)2^n a_2 = \alpha p_1,$$

$$2(1+2\lambda)a_3 3^n - (1+3\lambda)a_2^2 2^{2n} = \alpha p_2 + \frac{\alpha(\alpha-1)p_1^2}{2!},$$

$$-(1+\lambda)2^n a_2 = \alpha q_1,$$

and

$$2(1+2\lambda)(2a_2^2-a_3)3^n-(1+3\lambda)2^{2n}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)q_1^2}{2!}.$$

From (16) and (18)

$$p_1=-q_1.$$

Subtracting (19) from (17) and applying $p_1=-q_1$ the following are obtained

$$a_3=a_2^2+\frac{\alpha(p_2+q_2)}{4(1+2\lambda)3^n}. \tag{27}$$

On the other hand summing (17) and (19) we have

$$a_2^2(4(1+2\lambda)3^n-2(1+3\lambda)2^{2n})=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)(p_1^2+q_1^2)}{2!}, \tag{28}$$

from (28) and (21) we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{4\alpha(1+2\lambda)3^n-(2\alpha(1+3\lambda)+(\alpha-1)(1+2\lambda)^2)2^{2n}}. \tag{29}$$

From (27) and (29) it follows that

$$a_3-\mu a_2^2=\frac{\alpha}{4}\left[\left(h(\mu)+\frac{1}{(1+2\lambda)3^n}\right)p_2+\left(h(\mu)-\frac{1}{(1+2\lambda)3^n}\right)q_2\right],$$

where

$$h(\mu)=\frac{4\alpha(1-\mu)}{4\alpha(1+2\lambda)3^n-(2\alpha(1+3\lambda)+(\alpha-1)(1+2\lambda)^2)2^{2n}}.$$

Hence $$|a_3 - \mu a_2^2| \leq \begin{cases} \alpha|h(\mu)| & \text{for} |\mathrm{h}(\mu)| \geq \frac{1}{(1+2\lambda)3^{\mathrm{n}}} \\ \frac{\alpha}{(1+2\lambda)3^n} & \text{for} 0 \leq |\mathrm{h}(\mu)| \leq \frac{1}{(1+2\lambda)3^{\mathrm{n}}} \end{cases}.$$

COEFFICIENT BOUNDS AND FEKETE-SZEGO FUNCTIONAL ESTIMATE FOR THE CLASS $M_\Sigma^n(\lambda, \beta)$

Definition 3.1 A function $f(z)$ given by (1) is said to be in $M_\Sigma^n(\lambda, \beta)$ if the following condition are satisfied

$$f \in \Sigma,$$

$$Re\left\{(1-\lambda)\frac{D^{n+1}f(z)}{D^n f(z)} + \lambda \frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} > \beta \quad (30)$$

$$(0 \leq \beta < 1, \lambda \geq 0, n \in N_0 : z \in U, N_0 = N \cup \{0\})$$

and

$$Re\left\{(1-\lambda)\frac{D^{n+1}g(w)}{D^n g(w)} + \lambda \frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right\} > \beta \quad (31)$$

$$(0 \leq \beta < 1, \lambda \geq 0, n \in N_0 : w \in U)$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \quad (32)$$

The results in this section are as follows

Theorem 3.1 *Let $f(z)$ given by (1) be in the class $M_\Sigma^n(\lambda, \beta)$. then*

$$|a_2| = \sqrt{\frac{2(1-\beta)}{2(1+2\lambda)3^n - (1+3\lambda)2^{2n}}} \quad (33)$$

$$|a_3| = \frac{4(1-\beta)^2}{(1+\lambda)^2 2^{2n}} + \frac{(1-\beta)}{(1+2\lambda)3^n} \quad (34)$$

$$(0 \leq \beta < 1, \lambda \geq 0, n \in N_0).$$

Proof. It follows from (30) and (31) that there exist $p(z) \in P$ and $q(w) \in P$ such that

$$(1-\lambda)\left(\frac{D^{n+1}f(z)}{D^{n}f(z)}\right)+\lambda\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)=\beta+(1-\beta)p(z) \tag{35}$$

and

$$(1-\lambda)\left(\frac{D^{n+1}g(w)}{D^{n}g(w)}\right)+\lambda\left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)=\beta+(1-\beta)q(w). \tag{36}$$

where $p(z)$ and $q(w)$ have the form (14) and (15) respectively. Therefore

$$\beta+(1-\beta)p(z)=\beta+(1-\beta)(1+p_1z+p_2z^2+p_3z^3+\cdots)$$
$$=\beta+1+p_1z+p_2z^2+p_3z^3-\beta p_1z+\beta p_2z^2+\beta p_3z^3+\cdots$$
$$=1+(1-\beta)p_1z+(1-\beta)p_2z^2+\cdots$$

Also,

$\beta+(1-\beta)q(w)=1+(1-\beta)q_1w+(1-\beta)q_2w^2+\cdots.$

Equating the coefficients in (35) and (36) yields

$$(1+\lambda)2^n a_2=(1-\beta)p_1, \tag{37}$$

$$2(1+2\lambda)a_3 3^n-(1+3\lambda)2^{2n}a_2^2=(1-\beta)p_2, \tag{38}$$

$$-(1+\lambda)2^n a_2=(1-\beta)q_1, \tag{39}$$

and

$$2(1+2\lambda)(2a_2^2-a_3)3^n-(1+3\lambda)2^{2n}a_2^2=(1-\beta)q_2. \tag{40}$$

From (37) and (39) it follows that

$$p_1 = -q_1 \tag{41}$$

and

$$2(1+\lambda)^2 a_2^2 2^{2n} = (1-\beta)^2(p_1^2+q_1^2). \tag{42}$$

From (38) and (40) we have

$$4(1+2\lambda)a_2^2 3^n - 2(1+3\lambda)a_2^2 2^{2n} - (1-\beta)p_2 = (1-\beta)q_2.$$

A rearrangement together with the identity in (41) yields

$$a_2^2(4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}) = (1-\beta)(p_2+q_2).$$

Therefore, we have

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}}. \tag{43}$$

Applying Lemma (1.1) to the coefficients p_2 and q_2

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2(1+2\lambda)3^n - (1+3\lambda)2^{2n}}}.$$

Next, in other to find $|a_3|$, subtracting (40) from (38)

$$4(1+2\lambda)a_3 3^n - 4(1+2\lambda)a_2^2 3^n = (1-\beta)(p_2-q_2). \tag{44}$$

It follows from (41), (42) and (44) that

$$a_3 = \frac{(1-\beta)(p_2-q_2)}{4(1+2\lambda)3^n} + \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(1+\lambda)^2 2^{2n}}. \tag{45}$$

Applying Lemma 1.1 to the coefficients p_1, p_2, q_1 and q_2 in (45) yields

$$|a_3| \leq \frac{(1-\beta)}{(1+2\lambda)3^n} + \frac{4(1-\beta)^2}{(1+\lambda)^2 2^{2n}}$$

Putting $n = 0$ in theorem 3.1 we have

Corollary 3.1 *Let $f(z)$ given by (1) be in the class $M_\Sigma(\lambda, \beta)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+\lambda}}; \quad |a_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{1-\beta}{1+2\lambda},$$

which are the results obtained in Li and Wang [9].

Putting $\lambda = 0$ and $n = 0$ in theorem 3.1 we have

Corollary 3.2 *Let $f(z)$ given by (1) be in the class $S_\Sigma^{*\beta}$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}; \qquad |a_3| \leq 4(1-\beta)^2 + (1-\beta).$$

Putting $\lambda = 1$ and $n = 0$ in theorem 3.1 we have

Corollary 3.3 *Let $f(z)$ given by (1) be in the class $K_\Sigma(\beta)$. Then*

$$|a_2| \leq (1-\beta); \qquad |a_3| \leq (1-\beta)^2 + \frac{(1-\beta)}{3}.$$

In what follows, the Fekete-Szego functional estimate for the class $M_\Sigma^n(\lambda, \beta)$

Theorem 3.2 *Let $f(z)$ be of the form (1) be in $M_\Sigma^n(\lambda, \beta)$,then $\mu \in \mathbb{R}$ $0 \leq \beta < 1$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} (1-\beta)|h(\mu)| & \text{for} |\mathrm{h}(\mu)| \geq \frac{1}{(1+2\lambda)3^{\mathrm{n}}} \\ \frac{(1-\beta)}{(1+2\lambda)3^n} & \text{for} 0 \leq |\mathrm{h}(\mu)| \leq \frac{1}{(1+2\lambda)3^{\mathrm{n}}} \end{cases} \tag{46}$$

where

$$h(\mu) = \frac{4(1-\mu)}{4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}}.$$

Proof. From (37), (38), (39), (40) the following were obtained

$$(1+\lambda)2^n a_2 = (1-\beta)p_1,$$

$$2(1+2\lambda)a_3 3^n - (1+3\lambda)a_2^2 2^{2n} a_2^2 = (1-\beta)p_2,$$

$$-(1+\lambda)2^n a_2 = (1-\beta)q_1,$$

and

$$2(1+2\lambda)(2a_2^2 - a_3)3^n - (1+3\lambda)2^{2n} a_2^2 = (1-\beta)q_2$$

From (37) and (39)

$$p_1 = -q_1$$

and

$$\frac{2(1+\lambda)^2 a_2^2 2^{2n}}{(1-\beta)^2} = (p_1^2 + q_1^2).$$

Subtracting (40) from (38) and applying $p_1 = -q_1$ we have

$$a_3 = a_2^2 + \frac{(1-\beta)(p_2-q_2)}{4(1+2\lambda)3^n}. \tag{47}$$

On the other hand summing (40) and (38)

$$a_2^2(4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}) = (1-\beta)(p_2+q_2)$$

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}}. \tag{48}$$

From (47) and (48), we obtained

$$a_3 - \mu a_2^2 = \frac{(1-\beta)}{4}\left(\left(h(\mu) + \frac{1}{(1+2\lambda)3^n}\right)p_2 + \left((h(\mu) - \frac{1}{(1+2\lambda)3^n}\right)q_2\right)$$

where

$$h(\mu) = \frac{4(1-\mu)}{4(1+2\lambda)3^n - 2(1+3\lambda)2^{2n}}.$$

Hence $$|a_3 - \mu a_2^2| \le \begin{cases} (1-\beta)|h(\mu)| & \text{for} |\mathrm{h}(\mu)| \ge \frac{1}{(1+2\lambda)3^{\mathrm{n}}} \\ \frac{(1-\beta)}{(1+2\lambda)3^n} & \text{for} 0 \le |\mathrm{h}(\mu)| \le \frac{1}{(1+2\lambda)3^{\mathrm{n}}}. \end{cases}$$

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