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Abstract

An attempt is made to show how an important mathematical constant (namely π), in use for ages, is evaluated computers are used, no doubt, but what is the philosophy behind the program created to carry out the evaluation (to any specified degree of accuracy). The discussion focuses on what exactly is (or are) the cornerstone(s) of such programs.

Key words: *Pi, Gregory's Series, Maclaurin's series, Accuracy*

We take a look into how the well-known constant "π" is evaluate to some specified degree of accuracy. Any student in middle school first encounter π, just after the concept of a circle is introduced. A very natural question forms in the minds of these young learners: What is the area of a circle? Is there any formula that comes in handy? Teacher provide the answer by starting that the area of a circle is πr^2 . And it is at this stage that π is introduced as being a constant ratio that of the circumference of ANY circle to its diameter, the FIRST value assigned being $\frac{22}{7}$. (we later learnt that $\frac{355}{113}$ is also a good value to use for π, in calculations.

As a student moves through the middle school level π does not bother him anymore, and he is content with being able to solve elementary problems like what linear distance a circular wheel traverses when it rotates a specified number of times, its radius being given. There is a certain change in scenario when the student steps into the IXth standard and is introduced to the three systems of measuring angle in trigonometry.

When the circular system is discussed the student encounters conversion relation like π radians = 180 degrees

The student gets a little more serious about the non-recurring, non-terminating value of π just after completing high school, and before stepping into graduate level courses or during its progress. The series of James Gregory, on which the evaluation of π depends, creeps in later on and is stated below:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7} + \ldots.\ ad\ \inf \qquad (-\frac{\pi}{4} \le \theta \le \frac{\pi}{4})$$

(This may equivalent cast into the form to $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$infinity.)

The series unlike the Maclaurin series for $\tan\theta$ the agreement θ , express the argument in terms of its trigonometric tangent i.e. $\tan\theta$. If θ be replaced by $\frac{\pi}{4}$ and $\tan\theta$ replaced by 1, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$$

This series opens the gates to writing a suitable program to obtain the value of θ, to any desired degree of accuracy (more the number of terms taken from the series on the right, more accurate is the result).

This series has an inherent difficulty in that it converges very slowly. This is an indirect way of saying that (literally) thousands of terms have to be taken to give a value of π up to (say) eight or ten places of decimals. It is known for example, that about 80,000 to 1, 00,000 terms are needed for a value correct to four decimal places.

There is an avenue to overcome this difficulty, by using certain ALLIED SERIES, the most common being that due to Euler:

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}(1) = \frac{\pi}{4}$$

In other words

$$\pi = 4(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3})$$

$$= 4[\tan^{-1}\frac{1}{2}] + 4[\tan^{-1}\frac{1}{3}]$$

$$= 4[\frac{1}{2} - \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} - \ldots.] + 4[\frac{1}{3} - \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} - \ldots.]$$

It is easy to see that both the series in brackets because successive terms get smaller and smaller with appreciable rapidity.

One very rapidly converging series for π is that due to Machin (stated in 1706).

The series goes thus:

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

It is evident that the Gregory series for the second term on the right converges with formidable speed.

Formulas similar to the above are captured in the generalized Machin-like formulae, shown below:

$$c_0\frac{\pi}{4} = \sum_{n=1}^{N} c_n \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

here a_n and b_n are positive integer such that $a_n < b_n$ & c_n is a signed non-zero integer and c_0 is a positive integer.

William Rutherford [1-3] used the formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

It may be observed that this is just a variation of the Machin's formula stated earlier, the two signed terms in the end conveniently combining to $\tan^{-1}\frac{1}{239}$.In other words $\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99} = \tan^{-1}\frac{1}{239}$

Another very simple and important relation that enables to evaluate π is:

$$\sum_{r=1}^{r=\infty}\frac{1}{r^2} = \frac{\pi^2}{6},$$

which can be written as

$$6\sum_{r=1}^{r=\infty}\frac{1}{r^2} = \pi^2$$

Any program to evaluate π from this identify simply exploits the rule: take the sum of the squares of the reciprocals of the natural numbers upto and including N, a pre-specified natural

number (generally large, say about 1000 or more). Multiply the sum by 6, and take the square root of the result to obtain the value of π. As N is made larger and larger, the successive values of π that emerge are with greater and greater degrees of accuracy.

References

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