ON A NEW SUBCLASS OF UNIVALENT FUNCTION WITH NEGATIVE COEFFICIENTS USING A GENERALISED DIFFERENTIAL OPERATOR

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Abstract

We study a new subclass $T^{*}(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ of univalent functions with negative coefficients defined by Hadamard product using a generalized differential operator. Coefficient estimates, distortion theorems are established. Further, extremal properties and radii of close-to-convexity, starlikeness and convexity of the class $T^{*}(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ are obtained.

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

Which are analytic and univalent in the open disk $U = \{z : z \in \mathbb{C} |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$, See [3]. Let $T*(\varphi)$ and $K(\varphi)$, $(0 \leq \varphi < 1)$ denote the subclasses of functions in S. That is, starlike and convex functions of order φ respectively. Analytically, ($f \in T*(\varphi)$ if and only if, f is of the form (1.1) and satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (0 \leq \varphi < 1, \qquad z \in U)$$

Similarly, $f \in K(\varphi)$ if and only if, f is of the form (1.1) and satisfies

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}> 0, \qquad (0 \le \varphi < 1, z \in U)$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \ge 0, k \in N) \tag{1.2}$$

This class has introduced and studied by Silverman [9], Let $R^*(\varphi) = R \cap T^*(\varphi), CV(\varphi) = R \cap K * (\varphi)$. The classes $R^*(\varphi)$ and $K^*(\varphi)$ possess some interesting properties and have been extensively studied by Silverman [9] and others. The Hadamard Product (or Convolution) of two power series

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, g(z) = z - \sum_{k=2}^{\infty} b_k z^k \tag{1.3}$$

In T is defined by

$$(f * g)(z) = f(z) * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k \tag{1.4}$$

Furthermore, $f \in A$, Opoola [7] defined the differential operator $D^n(\mu,\beta,t)f(z) : A \to A$ such that

$$(D^0(\mu,\beta,t)f(z) = f(z)$$

$$D^1(\mu,\beta,t)f(z) = zD_t f(z) = ztf'(z) - z(\mu-\beta)t + (1+(\beta-u-1)t)f(z)$$

$$D^2(\varphi,\beta,t)f(z) = zD_t\big(zD_t f(z)\big) = zD_t\big(D(\mu,\beta,t)f(z)\big)$$

$$\vdots$$

$$D^n(\mu,\beta,t)f(z) = zD_t\big(D^{n-1}(\mu,\beta,t)f(z)\big), n \in N_0 = N \cup \{0\}$$

If $f(z)$ is given by (1.1), then from the above definition of $D^n(\mu,\beta,t)f(z)$ we have

(1.5)

$$D^n(\mu,\beta,t)f(z) = z + \sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k z^k$$

$0 \le \mu, \beta, t \ge 0$ and $n \in N_0 = N \cup \{0\}$

With the above differential operator, the convolution of two power series $f(z)$ and $g(z)$ is given by

$$D^n(\mu,\beta,t)(f*g)(z) = z - \sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n\, a_k b_k z^k \quad (1.6)$$

Motivated by Atshan et al. [2] and see [10] Now, we define a new subclass $T^*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ of the class T.

Definition 1.1. For $0 \le \varphi < 1, \alpha \ge 0$ and $0 \le \lambda \le \frac{1}{2}, 0 \le \mu \le \beta$, we let $T^*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ be the subclass of the class T consisting of functions of the form (1.2) and satisfying the analytic criterion.

$$Re\{\frac{zD^n(\mu,\beta,t)(f*g)'(z)}{D^n(\mu,\beta,t)(f*g)(z)} + \frac{\lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)}{D^n\,(\mu,\beta,t)(f*g)(z)} - \varphi\} \ge |\frac{zD^n(\mu,\beta,t)(f*g)'(z)}{D^n(\mu,\beta,t)(f*g)(z)} + \frac{D^n(\mu,\beta,t)(f*g)''(z)}{D^n(\mu,\beta,t)(f*g)(z)} - 1| \quad (1.7)$$

The main object of this paper is to study some geometric properties of the class $T^*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ like the coefficient bounds, extreme points, radii of starlikeness, convexity and close to convexity for the class $T^*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$. Furthermore, we obtain partial sums for this class. Atshan and Buti [1], Dziok and Srivastava [5], Goodman [6], Ruscheweyh [8], Silverman [9] studied the univalent functions for different classes.

2. COEFFICIENT INEQUALITIES

In the following theorem we obtain necessary sufficient condition for a function f to be in the class $T^*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$. We will mention some lemmas useful for our work.

Lemma 2.1. [8] Let $\varphi \ge 0$ and ω be any complex number. Then $Re(\omega) \ge \varphi$ if and if $|\omega - (1+\varphi)| < |\omega + (1-\varphi)|$.

Lemma 2.2. [8] Let $\alpha \geq 0, 0 \leq \varphi$ and $\theta \in \mathbb{R}$. Then $Re(\omega) > \alpha|w-1| + \varphi$ if and only if $Re\big(\omega\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta}\big) > \varphi$ where ω is a complex number.

Theorem 2.3. Let $f \in T$ be given by (1.2). The $f \in T*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ if and only if

$$\sum_{k=2}^{\infty}[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)k]^n a_k b_k \leq (1-\varphi) \qquad (2.1)$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)k]^n b_k}$$

Proof. Suppose $f \in T*(g,\varphi,\alpha,\lambda,\mu,\beta,t)$ and $|z| = 1$, then by definition (1.1)

$$Re\{\frac{zD^n(\mu,\beta,t)(f*g)'(z)}{D^n(\mu,\beta,t)(f*g)(z)} + \frac{\lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)}{D^n\,(\mu,\beta,t)(f*g)(z)} - \varphi\} \geq |\frac{zD^n(\mu,\beta,t)(f*g)'(z)}{D^n(\mu,\beta,t)(f*g)(z)} + \frac{D^n(\mu,\beta,t)(f*g)''(z)}{D^n(\mu,\beta,t)(f*g)(z)} - 1|$$

Using lemma (2), it suffices to show that

$$Re\{\left(\frac{zD^n(\mu,\beta,t)(f*g)'(z)}{D^n(\mu,\beta,t)(f*g)(z)} + \frac{\lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)}{D^n(\mu,\beta,t)(f*g)(z)}\right)\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta})\} > \varphi \qquad (2.2)$$

$$= Re\left[\frac{\big(zD^n(\mu,\beta,t)(f*g)'(z) + \lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)\big)\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D^n(\mu,\beta,t)(f*g)(z)}{D^n(\mu,\beta,t)(f*g)(z)}\right] > \varphi \;\; \theta \in \mathbb{R}$$

For convenience, let $A(z) = zD^n(\mu,\beta,t)(f*g)'(z) + \lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D^n(\mu,\beta,t)(f*g)(z)$ and $B(z) = D^n(\mu,\beta,t)(f*g)(z)$ that is equation (2.2) is equivalent to $Re(\frac{A(z)}{B(z)}) \geq \varphi$ Making use of lemma (2.1)

$$\left|\frac{A(z)}{B(z)} - (1+\varphi)\right| \leq \left|\frac{A(z)}{B(z)} + (1-\varphi)\right| \Longrightarrow \left|\frac{A(z)-(1+\varphi)B(z)}{B(z)}\right| < \left|\frac{A(z)+(1-\varphi)B(z)}{B(z)}\right|$$

$$|A(z) + (1-\varphi)B(z)| - |(A(z) - (1+\varphi)B(z)| \geq 0$$

Now

$$|A(z) + (1-\varphi)B(z)|$$

$$= \left|\left(\left(z - \sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n a_k b_k z^k\right) - \lambda\left(\sum_{k=2}^{\infty} k(k-1)[1 + (k+\beta-\mu-1)t]^n\, a_k b_k z^k\right)\right)\left(1+\alpha e^{i\theta}\right) - \alpha e^{i\theta}\left(\left(z - \sum_{k=2}^{\infty}[1 + (k+\beta-\mu-1)t]^n\, a_k b_k z^k\right) + (1-\varphi)(z - \sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n a_k b_k z^k)\right|$$

$$= |(z\left(1+\alpha e^{i\theta}\right) - \sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n \left(1+\alpha e^{i\theta}\right) a_k b_k z^k - \lambda(\sum_{k=2}^{\infty} k(k-1)[1 + (k+\beta-\mu-1)t]^n \left(1+\alpha e^{i\theta}\right) a_k b_k z^k - \alpha e^{i\theta} z + \sum_{k=2}^{\infty}[1 + (k+\beta-\mu-1)t]^n \alpha e^{i\theta} \alpha_k b_k z^k + (1-\varphi)z - (1-\varphi)\sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n a_k b_k z^k$$

$$= z + z\alpha e^{i\theta} - \sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n (1+\alpha e^{i\theta}) a_k b_k z^k - \lambda(\sum_{k=2}^{\infty}\sum_{k=2}^{\infty} k(k-1)[1 + (k+\beta-\mu-1)t]^n \left(1+\alpha e^{i\theta}\right) a_k b_k z^k - \alpha e^{i\theta} z + \sum_{k=2}^{\infty}[1 + (k+\beta-\mu-1)t]^n \alpha e^{i\theta} a_k b_k z^k + z - z\varphi - (1-\varphi)\sum_{k=2}^{\infty} k[1 + (k+\beta-\mu-1)t]^n a_k b_k z^k|$$

$$= |(2-\varphi)z - \sum_{k=2}^{\infty}[(k+\lambda k(k-1)\left(1+\alpha e^{i\theta}\right) - \alpha e^{i\theta} + (1-\varphi)][1 + (k+\beta-\mu-1)t]^n a_k b_k z^k|$$

$$\geq (2-\varphi)\sum_{k=2}^{\infty}[k(1+\lambda(k-1)(1-\alpha) - \alpha + (1-\varphi)][1+(k+\beta-\mu-1)t]^n a_k b_k$$

Also

$$
\begin{aligned}
&|A(z) + (1-\varphi)B(z)| \\
&= |\Bigg(\Bigg(z - \sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n a_k b_k z^k) \\
&- \lambda(\sum_{k=2}^{\infty} k(k-1)[1+(k+\beta-\mu-1)t]^n a_k b_k z^k\Bigg)\Bigg)\left(1+\alpha e^{i\theta}\right) \\
&- \alpha e^{i\theta}(\left(z - \sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k b_k z^k\right) - (1+\varphi)(z \\
&- \sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n a_k b_k z^k)| \\
&= |(z\left(1+\alpha e^{i\theta}\right) \\
&- \sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n\left(1+\alpha e^{i\theta}\right) a_k b_k z^k \\
&- \lambda(\sum_{k=2}^{\infty} k(k-1)[1+(k+\beta-\mu-1)t]^n\left(1+\alpha e^{i\theta}\right) a_k b_k z^k - \alpha e^{i\theta} z \\
&+ \sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n \alpha e^{i\theta} a_k b_k z^k + (1-\varphi) z \\
&- (1-\varphi)\sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n a_k b_k z^k \\
&= z + z\alpha e^{i\theta} \\
&- \sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n (1+\alpha e^{i\theta}) a_k b_k z^k \\
&- \lambda(\sum_{k=2}^{\infty}\sum_{k=2}^{\infty} k(k-1)[1+(k+\beta-\mu-1)t]^n\left(1+\alpha e^{i\theta}\right) a_k b_k z^k - \alpha e^{i\theta} z \\
&+ \sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n \alpha e^{i\theta} a_k b_k z^k - z - z\varphi \\
&+ (1+\varphi)\sum_{k=2}^{\infty} k[1+(k+\beta-\mu-1)t]^n a_k b_k z^k|
\end{aligned}
$$

$$
\begin{aligned}
&|\varphi z - \sum_{k=2}^{\infty}\left(\left(1+\alpha e^{i\theta}\right)\left(k + k\lambda(k-1)\right) - \alpha e^{i\theta} - (1+\varphi)\right)[1 \\
&\qquad + (k+\beta-\mu-1)t]^n a_k b_k z^k|
\end{aligned}
$$

Now with $|z| = 1$

$$\leq \varphi + \sum_{k=2}^{\infty} k\left((1+\alpha)\big(1+\lambda(k-1)\big) - \alpha - (1+\varphi)\right)[\,k[1+(k+\beta-\mu-1)t]^n a_k b_k$$

It is easy to show that

$$|A(z) + (1-\varphi)B(z)| - |A(z) - (1+\varphi)B(z)| = (2-\varphi)$$
$$-\sum_{k=2}^{\infty}[k(1+\lambda(k-1)(1-\alpha) - \alpha + (1-\varphi))[1+(k+\beta-\mu-1)t]^n a_k b_k$$
$$-\varphi - \sum_{k=2}^{\infty} k\left((1+\alpha)\big(1+\lambda(k-1)\big) - \alpha - (1+\varphi)\right)[1+(k+\beta-\mu-1)t^n a_k b_k$$

$$2(1-\varphi) - 2\sum_{k=2}^{\infty}\big(k(1+\alpha)(1+(k-1)\lambda) - (\alpha+\varphi)\big)[1+(k+\beta-\mu-1)t]^n a_k b_k \geq$$

$$\Rightarrow \sum_{k=2}^{\infty}\big(k(1+\alpha)(1+(k-1)\lambda) - (\alpha+\varphi)\big)[1+(k+\beta-\mu-1)t]^n a_k b_k \leq 1-\varphi$$

Conversely, suppose the inequality (2.1) holds, we need to show that

$$Re\left[\frac{\big(zD^n(\mu,\beta,t)(f*g)'(z) + \lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)\big)\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D^n(\mu,\beta,t)(f*g)(z)}{D^n(\mu,\beta,t)(f*g)(z)}\right]$$
$$\geq \varphi$$

then

$$Re\left[\frac{\big(zD^n(\mu,\beta,t)(f*g)'(z) + \lambda z^2 D^n(\mu,\beta,t)(f*g)''(z)\big)\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D^n(\mu,\beta,t)(f*g)(z) - \varphi}{D^n(\mu,\beta,t)(f*g)(z)}\right]$$
$$\geq$$

Or equivalently

$$Re\{\frac{((z-\sum_{k=2}^{\infty}k[1+(k+\beta-\mu-1)t]^{n}a_k b_k z^k)+\lambda(\sum_{k=2}^{\infty}k(k-1)[1+(k+\beta-\mu-1)t]^{n}a_k b_k z^k))\left(1+\alpha e^{i\theta}\right)-\alpha e^{i\theta}((z-\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t)}{z-\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n}a_k b_k z^k}$$

n

$$\Rightarrow Re\{\frac{z(1-\varphi)-\sum_{k=2}^{\infty}a_k b_k[1+(k+\beta-\mu-1)t]^{n}\left(k\left(1+\alpha e^{i\theta}\right)(1+(k-1)\lambda)-\left(\alpha e^{i\theta}+\varphi\right)\right)z^k}{z-\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n}a_k b_k z^k}\}$$
$$\geq 0$$

$$\Rightarrow Re\{\frac{z[(1-\varphi)-\sum_{k=2}^{\infty}a_k b_k\left[1+(k+\beta-\mu-1)t\right]^{n}\left(k\left(1+\alpha e^{i\theta}\right)(1+(k-1)\lambda)-\left(\alpha e^{i\theta}+\varphi\right)\right)z^{k-1}\right]}{z(1-\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n}\,a_k b_k z^{k-1})}\}$$
$$\geq 0$$

Since $\left|e^{i\theta}\right|=1$, Hence $Re(e^{i\varphi})\leq\left|e^{i\theta}\right|=1$, letting $|z|\to 1^{-1}$yields

$$\Rightarrow Re\{\frac{[(1-\varphi)-\sum_{k=2}^{\infty}a_k b_k z^k[1+(k+\beta-\mu-1)t]^{n}(k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi))]}{(1-\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^{n}\,a_k b_k)}\} \tag{2.3}$$
$$\geq 0$$

choosing the value of z to tend to 1 on the real axis that

$$\frac{[\left(zD^{n}(\mu,\beta,t)(f*g)'(z)+\lambda z^2 D^{n}(\mu,\beta,t)(f*g)''(z)\right)\left(1+\alpha e^{i\theta}\right)-\alpha e^{i\theta}D^{n}(\mu,\beta,t)(f*g)(z)-\varphi]}{D^{n}(\mu,\beta,t)(f*g)(z)}$$

is real. Upon clearing the denominator in (2.3) we obtain

$$\sum_{k=2}^{\infty}[a_k b_k[1+(k+\beta-\mu-1)t]^{n}(k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi))]]\leq 1-\varphi.$$

Finally, the result is sharp for the function

$$f(z) = z - \frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}$$

The proof is complete.

Corollary 2.4. Let$f \in T*(g,\varphi,\alpha,\mu,\beta,t)$. Then

$$a_k \leq \frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}$$

taking $\lambda = 0$ in theorem (2.3) we get the following corollary.

Corollary 2.5. *Let $f \in T$ given by (1.2). Then $f \in T*(g,\varphi,\alpha,\mu,\beta,t)$ if and only if*

$$\sum_{k=2}^{\infty}[k(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n \leq (1-\varphi)$$

3. **GROWTH THEOREM AND DISTORTION THEOREM**

Theorem 3.1. If $f \in T*(g,\varphi,\alpha,\mu,\beta,t)$ and $b_k \geq b_2$. Then

$$r - \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r^2 \leq |f(z)| \leq r + \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r^2$$

and

$$1 - \frac{2(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r \leq |f(z)| \leq |f'(z)| \leq 1 + \frac{2(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r$$

$(|z| = r < 1)$ *The result is sharp for*

$$f(z) = z - \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda) - (\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} z^2 \tag{3.1}$$

Proof. Since $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, we have

$$|f(z)| = |z - \sum_{k=2}^{\infty} a_k z^k| \leq |z| + \sum_{k=2}^{\infty} a_k |z|^k \leq r + r^2 \sum_{k=2}^{\infty} a_k \tag{3.2}$$

Since $k \geq 2$, $\big(2(1+\alpha)(1+\lambda) - (\alpha+\varphi)\big)[1+(\beta-\mu+1)t]^n b_2 \leq [k(1+\alpha)(1+(k-1)\lambda) - (\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k$
Using theorem (2.3) we have
$\big(2(1-\alpha)(1+\lambda) - (\alpha+\varphi)\big)[1+(\beta-\mu+1)t]^n b_2 \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} a_k b_k [k(1+\alpha)(1+(k-1)\lambda) - (\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n \leq 1-\varphi$
That is

$$\sum_{k=2}^{\infty} a_k \leq \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda) - (\alpha+\varphi)][1+(\beta-\mu+1)t]^n b_2}$$

Using the above equation in (3.2) we have,

$$|f(z)| \leq r + \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda) - (\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r^2$$

and

$$|f(z)| \geq r - \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda) - (\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_2} r^2$$

The result is sharp for

$$|f(z)| = z - \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda) - (\alpha+\varphi)](1+(\beta-\mu+1)t]^n b_n} z^2.$$

Similarly, since

$$f^{'}(z) = 1 - \sum_{k=2}^{\infty} k a_k z^{k-1}$$

we have that,

$$f^{'}(z) = |1 - \sum_{k=2}^{\infty} k a_k z^{k-1}| \leq |1| + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1 + 2r \sum_{k=2}^{\infty} a_k \tag{3.3}$$

Since for $n \geq 2$

$2(1+\alpha)(1+\lambda)-(\alpha+\varphi)[1+(\beta-\mu+1)t]^n b_2 \leq [k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k$

Using theorem (2.3) we have
$\big(2(1-\alpha)(1+\lambda)-(\alpha+\varphi)\big)[1+(\beta-\mu+1)t]^n b_2 \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} a_k b_k [k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n \leq 1-\varphi$
That is

$$\sum_{k=2}^{\infty} a_k \leq \frac{(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)][1+(\beta-\mu+1)t]^n b_2}$$

Using the above equation in (3.3) we have,

$$|f'(z)| \leq 1+\frac{2(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)][1+(\beta-\mu+1)t]^n b_2} r$$

and

$$|f'(z)| \geq 1-\frac{2(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)][1+(\beta-\mu+1)t]^n b_2} r$$

The result is sharp for

$$|f'(z)| = 1-\frac{2(1-\varphi)}{[2(1+\alpha)(1+\lambda)-(\alpha+\varphi)][1+(\beta-\mu+1)t]^n b_n} z.$$

This completes the proof.

4. EXTREME POINT

Theorem 4.1. Let $f_1(z)=z$, then

$$f_k(z) = z-\frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} z^k \tag{4.1}$$

where $(k\in\mathbb{N}, 0\leq\varphi<1, \alpha\geq 0)$.

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z) \tag{4.2}$$

where $[\mu_k \geq 0, \sum_{k=1}^{\infty}\mu_k = 1 \; or \; 1=\mu_1+\sum_{k=2}^{\infty}\mu_k]$

Proof. $f(z)$ can be expressed as in (4.2) as

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)$$

$$= \mu_1 f_1(z) + \sum_{k=1}^{\infty} \mu_k f_k(z)$$

$= \mu_1 z + \sum_{k=2}^{\infty} \mu_k \{z - \frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} z^k\}$

$(\mu_1 + \sum_{k=2}^{\infty} \mu_k) z - \sum_{k=2}^{\infty} \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} z^k\}$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} z^k$$

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

where

$$a_k = \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}.$$

Therefore $f \in T^*(g, \alpha, \varphi, \lambda, \mu, \beta, t)$ since from theorem 2.3 that

$$\sum_{k=2}^{\infty} a_k b_k \frac{k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)[1+(k+\beta-\mu-1)t]^n}{(1-\varphi)} \leq 1$$

Hence,

$\sum_{k=2}^{\infty} \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} X b_k \frac{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n}{(1-\varphi)}$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 < 1$$

so by theorem 2.3 $f \in T^*(g, \alpha, \varphi, \lambda, \mu, \beta, t)$. Conversely, suppose$f \in T *$ $(g, \alpha, \varphi, \lambda, \mu, \beta, t)$, then by equation (2.1), we may set

$$a_k = \frac{(1-\varphi)}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}, (k \geq 2)$$

we set

$$\mu_k = \frac{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(1-\varphi)}$$

and $\mu_1 = 1 - \sum_{k=1}^{\infty} \mu_k$,
then $f(z) = z - \sum_{k=2}^{\infty} \alpha_k z^k$

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} z^k$$

(4.3)

Therefore,

$$z^k = \frac{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t^n}{(1-\varphi)} (z - f_k(z))$$

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(1-\varphi)\mu_k}{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k} \text{x} \frac{[k(1+\alpha)(1+(k-1)\lambda)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n}{(1-\varphi)} (z - f_k(z))$$

$$f(z) = z - \sum_{k=2}^{\infty} \mu_k z + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= z(1 - \sum_{k=2}^{\infty} \mu_k) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= z\mu_1 + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z)$$

5

5. RADII OF UNIVALENT STARLIKENESS, CONVEXITY AND CLOSE TO CONVEXITY

Theorem 5.1. If $f \in T^*(g,\varphi,\alpha,\mu,\beta,t)$ then f is starlike of order $\delta (0 \le \delta < 1)$ in the disc $|z| < r_1(g,\varphi,\alpha,\mu,\beta,t,\delta)$, where

$$r_1(g,\varphi,\alpha,\mu,\beta,t,\delta) = Inf_k \{\frac{(1-\delta)[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(k-\delta)(1-\varphi)}\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{5.1}$$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \delta \qquad (0 \le \delta < 1)$$

for $|z| < r_1(g,\varphi,\alpha,\mu,\beta,t,\delta)$ we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{z(1 - \sum_{k=2}^{\infty} k a_k z^{k-1})}{z(1 - \sum_{k=2}^{\infty} a_k z^{k-1})} - 1\right| = \left|\frac{(1 - \sum_{k=2}^{\infty} k a_k z^{k-1})}{(1 - \sum_{k=2}^{\infty} a_k z^{k-1})} - 1\right|$$

$$= \left|\frac{(1 - \sum_{k=2}^{\infty} k a_k z^{k-1}) - (1 - \sum_{k=2}^{\infty} a_k z^{k-1})}{(1 - \sum_{k=2}^{\infty} a_k z^{k-1})}\right|$$

$$\left|\frac{-\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{(1-\sum_{k=2}^{\infty}a_k z^{k-1})}\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}} \leq 1-\delta. \tag{5.2}$$

If

$$\sum_{k=2}^{\infty}\frac{(k-\delta)a_k|z|^{k-1}}{1-\delta} \leq 1$$

Hence by theorem (2.3), (5.2) will by true if

$\frac{(k-\delta)a_k|z|^{k-1}}{1-\delta} \leq \frac{[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(1-\varphi)}$

and hence,

$|z| \leq \{\frac{(1-\delta)[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(k-\delta)(1-\varphi)}\}^{\frac{1}{k-1}}$

Setting $|z| = r_1(g,\varphi,\alpha,\mu,\beta,t,\delta)$ we get the desired result.

Theorem 5.2. If $f \in T^*(g,\varphi,\alpha,\mu,\beta,t)$ then f is convex of order $\delta(0 \leq \delta < 1)$ in the disc $|z| < r_2(g,\varphi,\alpha,\mu,\beta,t,\delta)$, where

$r_2(g,\varphi,\alpha,\mu,\beta,t,\delta) = Inf_k\{\frac{(1-\delta)[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{k(k-\delta)(1-\varphi)}\}^{\frac{1}{k-1}} \quad (k \geq 2).$ (5.3)

Proof. It is sufficient to show that

$\left|\frac{zf''(z)}{f'(z)}+1-1\right| = \left|\frac{\sum_{k=2}^{\infty}(k(k-1)a_k z^{k-1})}{1-\sum_{k=2}^{\infty}ka_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}ka_k|z|^{k-1}} \leq 1-\delta$

then

$$\sum_{k=2}^{\infty}\frac{k(k-\delta)a_k|z|^{k-1}}{1-\delta} \leq 1-\delta \tag{5.4}$$

Hence, by theorem (2.3), (5.4) will be true if

$$\frac{k(k-\delta)|z|^{k-1}}{1-\delta} \leq \frac{1}{\sum_{k=2}^{\infty}a_k}$$

$$\frac{k(k-\delta)|z|^{k-1}}{1-\delta} \leq \left\{\frac{[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(1-\varphi)}\right\}$$

and hence

$|z| \leq \{\frac{(1-\delta)[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(1-\varphi)}\}^{\frac{1}{k-1}}$

Setting $|z| = r_2(g,\varphi,\alpha,\mu,\beta,t,\delta)$ we get the desire result.

Theorem 5.3. Let a function $f \in T^*(g,\varphi,\alpha,\mu,\beta,t,\delta)$. Then f is close to convex of order $\delta(0 \leq \delta < 1)$ in the disc $|z| \leq r_3(g,\varphi,\alpha,\mu,\beta,t,\delta)$ where

$$r_3(g,\varphi,\alpha,\mu,\beta,t,\delta) = Inf_k\{\frac{(1-\delta)[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{k(1-\varphi)}\}^{\frac{1}{k-1}} \quad (k \geq 2). \qquad 5.5$$

Proof. It is sufficient to show that

$$|f'(z)-1| \leq 1-\delta \qquad (0 \leq \delta < 1)$$

for $|z| \leq r_3(g,\varphi,\alpha,\mu,\beta,t,\delta)$ we have,

$$|f'(z)-1| = \left|1-\sum_{k=2}^{\infty} ka_k z^{k-1} - 1\right| = \left|-\sum_{k=2}^{\infty} ka_k z^{k-1}\right| \leq \sum_{k=2}^{\infty} ka_k |z|^{k-1} \leq 1-\delta$$

If

$$\sum_{k=2}^{\infty} \frac{ka_k|z|^{k-1}}{1-\delta} \leq 1 \qquad 5.6$$

Then by Theorem (2.3), (5.6) will be true if

$$\frac{ka_k|z|^{k-1}}{1-\delta} \leq \frac{[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{(1-\varphi)}$$

and hence

$$|z| \leq \{\frac{[k(1+(k-1)\lambda)(1+\alpha)-(\alpha+\varphi)][1+(k+\beta-\mu-1)t]^n b_k}{k(1-\varphi)}\}^{\frac{1}{k-1}} \qquad (k \geq 2).$$

Setting $|z| = r_3(g,\varphi,\alpha,\mu,\beta,t,\delta)$ we get the desired result. The proof is complete.

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