A REVIEW ON NON-UNIQUE REPRESENTATIONS OF POSITIVE INTEGERS AS $2^X + 3^Y$ FOR NON-NEGATIVE INTEGERS X AND Y

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Abstract

We have reviewed the proof[1] of a conjecture posted in the Online Encyclopedia of Integer Sequences (OEIS)[2], which states that there are exactly five positive integers that can be represented in more than one way as the sum of non-negative integral powers of 2 and 3. The case for both powers being positive follows from a theorem of Bennett. We have also devised alternative logics to the elementary methods applied in the original paper "On duplicate representations as 2^x+3^y for nonnegative integers x and y"[1] to prove the case where zero exponents are allowed. Besides, we have included a nominal program towards the end, that verifies the statement of the conjecture.

Keywords: *Online Encyclopedia of Integer Sequences (OEIS), exactly five positive integers, non-negative powers of 2 and 3, Bennett theorem and zero exponents.*

INTRODUCTION

The Online Encyclopedia of Integer Sequences (OEIS) encases a sequence, namely A004050[3], comprising integers of the form $2^x + 3^y$ for non-negative integers x and y. The sequence, on its entry in the OEIS, was remarked as a conjecture in September 2012 that only five of these integers can be so expressed in two different ways.

In fact, sequence A085634 lists those very integers expressible both as $2^x + 3^y$ and $2^a + 3^b$, where x, y, a, and b non-negative integers and x > a. The five elements listed are as follows:

a) $5 = 2^2 + 3^0$, $5 = 2^1 + 3^1$

b) $11 = 2^3 + 3^1$, $11 = 2^1 + 3^2$

c) $17 = 2^4 + 3^0$, $17 = 2^3 + 3^2$

d) $35 = 2^5 + 3^1$, $35 = 2^3 + 3^3$

e) $259 = 2^8 + 3^1$, $259 = 2^4 + 3^5$

On the entry of sequence A085634[4] in the OEIS, it was remarked in February 2005 that if n belongs to the sequence and n > 259, then n > 10^{4000}. In this note, this lower bound is rendered vacuously true by proving the conjecture that the five numbers listed above are the only elements of A085634.

We thus assume that:

$$2^x + 3^y = 2^a + 3^b \tag{1}$$

where x, y, a, and b are non-negative integers, such that (without loss of generality) x > a (whence y < b). Equivalently,

$$2^x - 3^b = 2^a - 3^y \tag{2}$$

This brings us to sequence A207079[5] in the OEIS, which is described in its entry as "the only non-unique differences between powers of 3 and 2." It is given as a finite sequence of five elements, namely 1, 5, 7, 13 and 23. It is commented that the finiteness of this sequence is due to Bennett [6]. The finiteness of the specific sequence A207079 was first proved in 1982. Here, it is stated as a lemma, the special case of Bennett's result that applies most directly to (2).

Lemma 1 (Bennett): There are precisely three integers of the form 2^x-3^b, with x and b natural numbers, that are also expressible as 2^a-3^y, with a and y natural numbers such that x > a. They are

$-1 = 2^3 - 3^2 = 2^1 - 3^1$, $\qquad 5 = 2^5 - 3^3 = 2^3 - 3^1$, $\qquad 13 = 2^8 - 3^5 = 2^4 - 3^1$.

These are, respectively, the only two such representations for these three integers. All other integers have either a unique such representation, or none at all.

Bennett's result is applied to the cases of (1) and (2) where x, y, a, and b are all positive integers. This leaves us with the special case when y = 0; clearly (1) and (2) are impossible if a = 0.The special case y = 0 is proved by elementary methods, except for the one instance where Lemma 1 is applied to deduce that 1 has only the single representation $1 = 2^2 - 3^1$.

ANALYSIS

CASE 1: when y > 0

Lemma 2: There are precisely three solutions to (1) when y > 0. They are

$11 = 2^3 + 3^1 = 2^1 + 3^2$,

$35 = 2^5 + 3^1 = 2^3 + 3^3$,

$259 = 2^8 + 3^1 = 2^4 + 3^5$.

Proof:

Let $c = 2^a - 3^y$ in (2).

By Lemma 1, if $c \notin \{-1,5,13\}$, then $x = a$, which contradicts the hypothesis $x > a$. Else, $c \in \{-1,5,13\}$.

Let $c = -1$.

By Lemma 1, two representations are possible, as in (2): $-1 = 2^3 - 3^2 = 2^1 - 3^1$.

Thus, $x = 3$, $b = 2$, $a = 1$, and $y = 1$.

This gives: $2^3 + 3^1 = 2^1 + 3^2 = 11$.

Let $c = 5$.

Then similarly as above, we get: $5 = 2^5 - 3^3 = 2^3 - 3^1$,

which gives: $2^5 + 3^1 = 2^3 + 3^3 = 35$.

Let $c = 13$.

Then similarly as above, we get: $13 = 2^8 - 3^5 = 2^4 - 3^1$,

which gives: $2^8 + 3^1 = 2^4 + 3^5 = 259$.

CASE 2: when $y = 0$

For a prime p and a natural number n, it is written $p \parallel n$ if $p \mid n$ but $p^2 \nmid n$. Here, the p-valuation of n is denoted by $v_p(n)$; where $v_p(n) = k$ if $p^k \parallel n$.

Lemma 3: If n is a natural number then:

$$v_2(3^n - 1) = \begin{cases} 1, & \text{if } 2 \nmid n; \\ 2 + v_2(n) & \text{if } 2 \mid n. \end{cases}$$

Lemma 4: If n is a natural number then:

$$v_3(2^n - 1) = \begin{cases} 0, & \text{if } 2 \nmid n; \\ 1 + v_3(n) & \text{if } 2 \mid n. \end{cases}$$

Lemma 3 and 4 follow easily from Basic Mathematics applying concepts of Binomial Theorem and Inductive reasoning as shown below:

Proof:

(I) When $2 \nmid n$, let $n = 2k + 1$ ($k \in I^+$)

Now, $3^{2k+1} - 1 = 3.9^k - 1 = 3(1 + {}^kC_1 .8 + {}^kC_2 .8^2 + \ldots\ldots\ldots + {}^kC_{k-1} .8^{k-1} + 8^k) - 1$

$= 3\text{-}1 + 3.\ {}^kC_1 .8 + 3.{}^kC_2 .8^2 + \ldots\ldots\ldots + 3.\ {}^kC_{k-1} .8^{k-1} + 3.\ 8^k$

$= 2 + 2N$ ($N \in I^+$)

$= 2(1+N)$

$= 2^1$ x (an odd integer)

So, $v_2(3^n - 1) = v_2(2^1$ x (an odd integer)) = 1 (by definition of $v_p(n)$)

When $2 \mid n$, we observe the following pattern for $3^n - 1$:

For n=2 : $3^2 - 1 = 9 - 1 = 8 = 2^3$; so $v_2(3^2 - 1) = 3$ (using definition of $v_p(n)$)

$= 2 + 1 = 2 + v_2(2)$

For n=4: $3^4 - 1 = 81 - 1 = 80 = 2^4.\ 5^1$; so $v_2(3^4 - 1) = 4$ (using definition of $v_p(n)$)

$= 2 + 2 = 2 + v_2(4)$

For n=6: $3^6 - 1 = 729 - 1 = 728 = 2^3.7^1.13^1$; so $v_2(3^6 - 1) = 3$ (using definition of $v_p(n)$)

$= 2 + 1 = 2 + v_2(6)$

Following this pattern, we must get that for n=2k ($k \in I^+$): $v_2(3^n - 1) = 2 + v_2(n)$

(II) When $2 \nmid n$, let $n = 2k + 1$ ($k \in I^+$)

Now, $2^{2k+1} - 1 = 2.4^k - 1 = 2(1+3)^k - 1 = 2(1 + {}^kC_1 .3 + {}^kC_2 .3^2 + \ldots. + {}^kC_{k-1} .3^{k-1} + 3^k) - 1$ 2-1 + $2.\ {}^kC_1 .3 + 2.{}^kC_2 .3^2 + \ldots\ldots\ldots + 2.\ {}^kC_{k-1} .3^{k-1} + 2.\ 3^k$

$= 1 + 3N$ ($N \in I^+$) which is not a multiple of 3

Therefore, $v_3(2^n - 1) = v_3(1 + 3N) = 0$

When $2 \mid n$, we observe the following pattern for $2^n - 1$:

For n=2 : $2^2 – 1 = 4 – 1 = 3 = 3^1$; so $v_3(2^2 – 1) = 1$ (by definition of $v_p(n)$)

$= 1 + 0 = 1 + v_3(2)$

For n=4 : $2^4 – 1 = 16 – 1 = 15 = 3^1. 5^1$; so $v_3(2^4 – 1) = 1$ (by definition of $v_p(n)$)

$= 1 + 0 = 1 + v_3(4)$

For n=6 : $2^6 – 1 = 64 – 1 = 63 = 3^2.7^1$; so $v_3(2^6 – 1) = 2$ (by definition of $v_p(n)$)

$= 1 + 1 = 1 + v_3(6)$

Following this pattern, we must get that for n=2k ($k \in I^+$): $v_3(2^n – 1) = 1 + v_3(n)$

Lemma 5: There are precisely two solutions to (1) when y = 0. They are

$5 = 2^2 + 1 = 2^1 + 3^1$, $17 = 2^4 + 1 = 2^3 + 3^2$.

Proof: We are given:

$$2^x + 1 = 2^a + 3^b, \tag{3}$$

where x, a, and b are natural numbers, where x > a. Let s = x − a. Thus,

$$2^a(2^s - 1) = 3^b - 1. \tag{4}$$

It is necessary by Lemma 4 that s is odd, as, by (4), $3 \nmid 2^s - 1$.

First, suppose b is odd; then Lemma 3 implies $2 \parallel 3^b - 1$, hence, by (4), a = 1.

Thus, by (3),

$2^x - 3^b = 1$.

Thus, by Lemma 1, x = 2 and b = 1. This produces the equation: $2^2 + 1 = 2 + 3 = 5$.

It remains to let b be even. Then $a = 2 + v_2(b)$ by Lemma 3. Suppose $2^2 \mid b$. Then $3^4 - 1 \mid 3^b - 1$, hence $5 \mid 3^b - 1$. Then (4) implies $5 \mid 2^s - 1$, hence $4 \mid s$, a contradiction as s is odd. Therefore $2 \parallel b$ and a = 3.

Writing b = 2c for an odd natural number c, we have by (4):

$$2^s - 1 = \frac{3^c - 1}{2} \cdot \frac{3^c + 1}{4}$$

Letting

$$z = \frac{3^c + 1}{4}$$

then z is a natural number by Lemma 3, and we obtain the quadratic in z:

$2^s - 1 = 2z^2 - z.$

Completing the square yields:

$(4z - 1)^2 = 2^{s+3} - 7.$

Writing s = 2t + 1 yields the difference of squares factorization:

$(2^{t+2} - 4z + 1)(2^{t+2} + 4z - 1) = 7.$

Therefore, $2^{t+2} - 4z + 1 = 1, \quad 2^{t+2} + 4z - 1 = 7;$

thus, $2^{t+2} = 4z = 4.$

Therefore t = 0, z = 1; thus, c = 1.

Hence s = 1 and b = 2.

Recalling a = 3, we have x = 4.

This gives the equation: $2^4 + 1 = 2^3 + 3^2 = 17.$

An alternative approach:

We have already proved that $4 \nmid b$ but $2 \mid b$. So, b=2m ($m \in I^+$)

Let m=2k ($k \in I^+$) (i.e. m be even), then b=4k i.e. $4 \mid b$ which contradicts $4 \nmid b$.

So, m can't be even.

Rewriting the right hand side expression of (4) in terms of m, we get $3^{2m} – 1 = 9^m – 1$

Now, we observe :

$9^1 – 1 = 8 = 8 \times 1 = 8 \times$ (an odd integer)

$9^3 – 1 = 728 = 8 \times 91 = 8 \times$ (an odd integer)

$9^5 – 1 = 59048 = 8 \times 7381 = 8 \times$ (an odd integer)

And so on.

Let us assume $(9^p - 1)$ gives 8 x (an odd integer) for p being an odd integer

Thus, we can write: $9^p - 1 = 8$ x (an odd integer) $= 8 \times (2k + 1)$; $(k \in I^+) = 16k + 8$

Getting $9^p = (16k + 9)$ (i)

Now, for the next odd integral power of 9 i.e. p+2,

$9^{p+2} - 1 = 81 \times 9^p - 1 = 81 \times (16k + 9) - 1$ (using (i))

$= 1296k + 728$

$= 8(128k + 91)$

$= 8(2(64k + 45) + 1)$

= 8(even + 1)

= 8 x odd (ii)

Concluding by mathematical induction, from (i) and (ii), $9^m - 1 = 8$ x (an odd integer) $\forall$ m; m=1,3,5,7,…….(2k-1)

So, right hand side of (4) = 8 x (an odd integer)

Therefore, left hand side should also be 8 x (an odd integer)

i.e. $2^a(2^s - 1) = 8$ x (an odd integer)

but an odd integer is not divisible by any power of 2

So, $2^a = 8$ (Since 2^s-1 is an odd integer)

i.e. a=3

Hence, from (3), we get $2^x + 1 = 2^3 + 3^b$ i.e. $2^x - 3^b = 2^3 - 1 = 7 = 16 - 9 = 2^4 - 3^2$

i.e. x=4, b=2 (using lemma 1)

So, $2^x + 1 = 2^4 + 1 = 17 = 2^3 + 3^2 = 2^a + 3^b$.

VERIFICATION

Besides reviewing the theoretical proof, we have devised a very simple program whose outcome supports this conjecture. The range of input can be taken as desired and the numbers within that range belonging to the sequence (if at all present) will be displayed. The program was tested(run) against input ranging from 2 to 9223372036854775807 (the largest integer a data type can hold[7] (we couldn't reach the value 10^{4000} due to size limitations)). But this problem can be eradicated if we use more powerful computers) and the results were only these five integers: 5, 11, 17, 35 and 259.

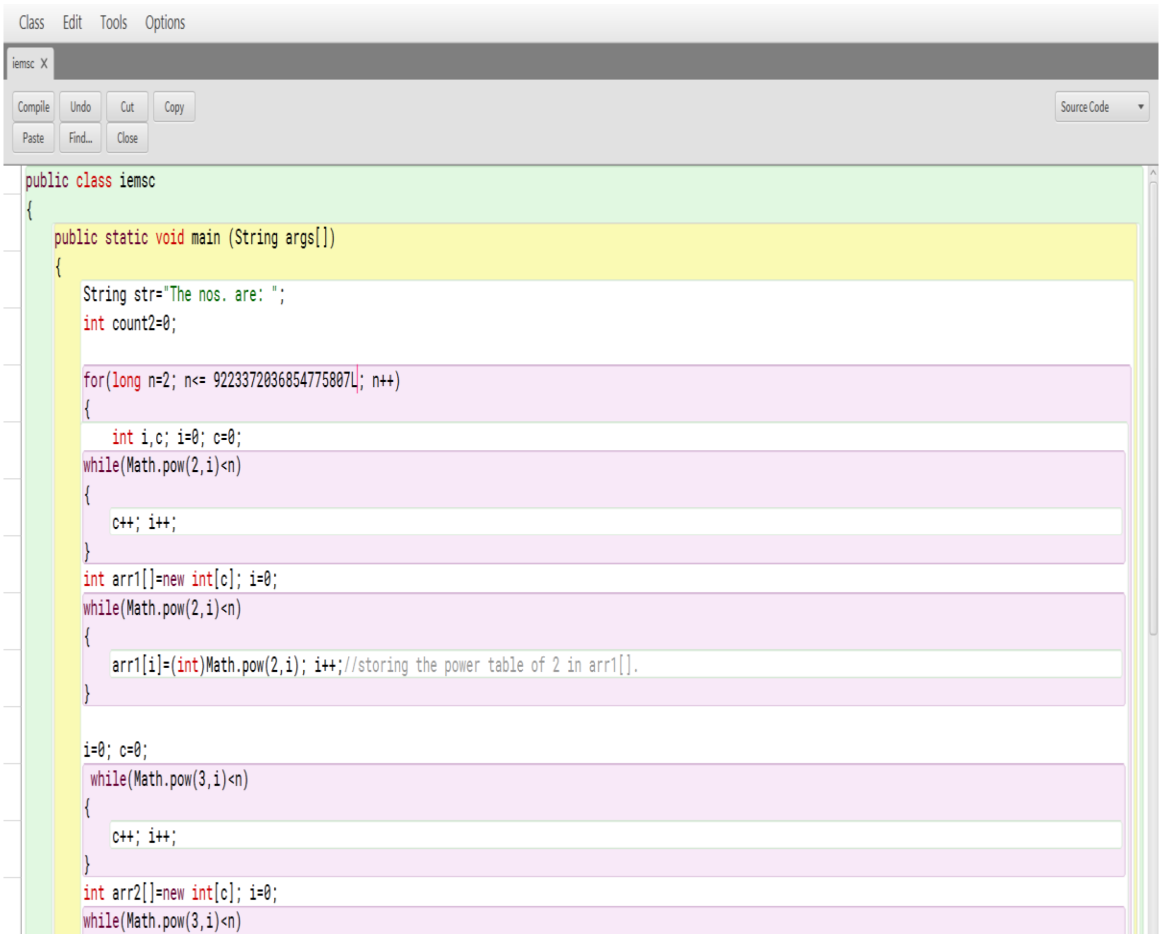

Fig1 Program code (in JAVA), for printing all the integers from 2 to 9223372036854775807, that can be be represented as $2^x + 3^y$ in more than one way (two ways).

Fig.2 Program code (in JAVA), for printing all the integers from 2 to 9223372036854775807, that can be be represented as $2^x + 3^y$ in more than one way (two ways).

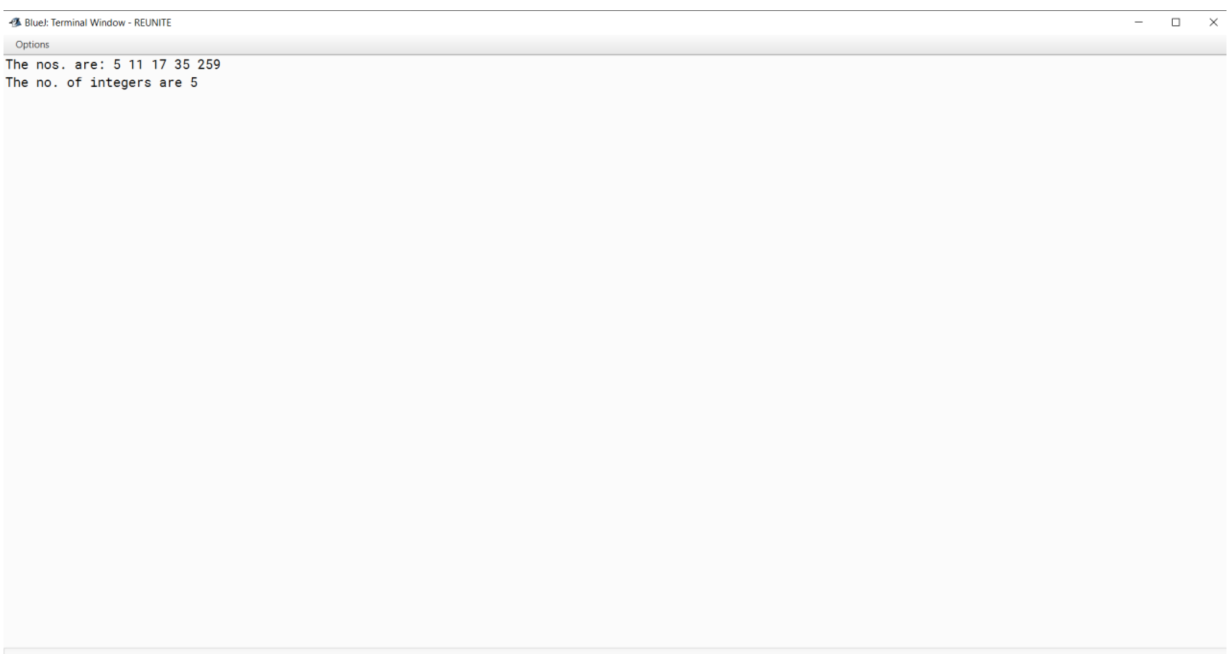

Fig 3 Output of the program after compiling successfully

With a little modification of the code, it can also be shown that the number of pairs (x,y) satisfying the equation: $a^x - b^y = c$ (where a, b, c, x and y are positive integers and a, b ≥ 2) will never exceed '2' which is in compliance with the Bennett theorem.

CONCLUSION

The conjecture that there are exactly five positive integers: 5, 11, 17, 35 and 259 that can be written in more than one way as the sum of a non-negative power of 2 and a non-negative power of 3 is now verified to be true. This fact itself is quite surprising that despite so many positive integers existing within 10^{4000}, only five such integers satisfy the above criterion. Throughout this paper, only elementary methods are used. Attempts like this that involve verification and/or justification of members of a number sequence, improves the understanding of similar number patterns and help us to analyze how they behave when subjected to specific mathematical conditions.

ACKNOWLEDGEMENT

We, the authors of this paper, do hereby acknowledge the inspiration and support rendered to us by the respected professors of the Mathematics department of Institute of Engineering & Management (IEM). Without their experience and guidance, it would have been impossible for us to write this paper flawlessly. We are grateful to everyone whose helpful contributions have been instrumental in the successful upbringing of this paper.

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