FURTHER ON KULKARNI'S AND RELATED UNIFIED METHODS FOR SOLVING GENERALPOLYNOMIAL EQUATIONS OF DEGREE LESS THAN FIVE

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Abstract

For $n \in \mathbb{N}$ a positive integer, let $f_n(x) = \sum_{j=0}^{n} a_{n-j}x^{n-j}\,; a_n \neq 0; n < 5$ be a general polynomial equation of degree less than five in one variable. Before now, we are unaware of any known unified method for solving $f_n(x) = 0$ $(n < 5)$ in radical until Kulkarni did so in 2006. However, it is quite ironical and contradicting to observe (one among many other issues) that the method introduced by Kulkarni fails if $n = 1$ (since $n = 1$ satisfies $n < 5$) which is the case of general polynomial equation of degree one. It is the purpose of this paper to address this issue and many more with an approach that is quite simple and novel.

Keywords: *Quartic equation, Cubic equation, resolvent equation. Auxiliary function, seeking solution.*

1. INTRODUCTION

Kulkarni asked a pertinent question in 2006 and I quote "Is there some common or unified approach for solving general polynomial equations of degree four or less instead of solving them by different methods?" Kulkarni (2006). The answer was given partially in the affirmative sense by Kulkarni since the method he proposed is not valid (collapse) for (solving general) polynomial equations of degree one. However, we observe that, it may probably be that Kulkarni is unaware that his question have been answered partially by Kalman and White (2001) whose method is still not valid for $n = 1$, they provided a beautiful unified approach, based on circulant matrices, for solving polynomial equations of degree four or less.

In this paper we investigate the approach proposed by Kulkarni and further suggest an alternative method of a unified approach which is quite simple and interesting for solving general polynomial equations of degree less than five and also remains valid even for polynomials of degree one, unlike that of Kalman and White (2001) and Kulkarni (2006) which collapses for polynomials of degree one.

We begin by presenting succinctly the work done by some mathematician in this field of study, the solution of some degree of polynomial equations. As far back as fifteenth centuries research has been going on in this area, for finding the solution of polynomial equations. Indeed literature indicates that polynomial equations have been investigated for centuries. Linear and quadratic polynomial equations were solved in the fifteenth centuries while that of cubic and quartic polynomial equations were comprehensively solved in the sixteenth centuries.

Cubic equation was known since the ancient times, even by the ancient Greeks and the ancient Babylonians and also the ancient Egyptians, who dealt with the problem of doubling the cubic.

In the 11th century, the famous Mathematician *Omar Khayyam* discovered a geometrical method to solve cubic equation which could be used to get numerical answer by intersecting a parabola with a circle, and by using this method he found cubic equation can have more than one solution. He could not find algebraic formula for the general but he could only solve cubic equation geometrically.

In 1515 Scipione del Ferro discovered a formula that solved the so called “depressed cubic”. Instead of publishing his solution, Del Ferro kept it a secret until his deathbed telling his student Antonio Fior. Niccolo Fotana known as Tartaglia (1500-1557) wins a mathematical contest by solving many different cubic, and gives his method to Cardan. In 1539 Girolamo Cardan (1501-1576) gives complete solution of cubic in his book, *The Great Art, or the Rules of Algebra.* In that book *the Ars magma,* Cardano introduced the technique of substitution by Ludovico Ferrari (1522-1565) that not only solved the cubic and quartic but became indispensable in polynomial algebra.

It may interest you to know that the first attempt to unify solutions to quadratic, cubic and quartic equations date at least to Lagrange (1869). Lagrange's analysis characterized the general solutions of the cubic and quartic cases in terms of permutations of the roots, laying a foundation for the independent demonstrations by Abel and Galois of the impossibility of solutions by radicals for general 5^{th} degree or higher equations. Since then researcher have pitch their tent in providing several other alternative (analytic) methods for the solution of general polynomial equation of degree less than five has been proposed in the literature. As a way of contribution to the existing unified methods, In this paper we present yet another alternative approach to the solution of a general polynomial equations of degree less than five.

Kulkarni (2006), introduced unified method for solving general polynomial equations of degree less than five by seeking for the zero solution of the polynomial (auxiliary function) of degree $n\ (n < 5)$ given by

$$g(x; b_0, \cdots b_{m-1}, c_0,, \cdots c_{m-1}, p) = \frac{(V_m(x))^k - p^k (W_m(x))^k}{1-p^k} - - - (1.1)$$

where $V_m(x)$ and $V_m(x)$ are constituent polynomials of degree m, such that $m < n$ and p is unknown to be determined. The integer k has to satisfy the relation $km = n$ so that the auxiliary function will be of degree n. It is important to observe that in this method, when n is even, the number of unknowns is one more than the number of equations, this is an issue to worry about. However, in such case, Kulkarni remarked and I quote "extra (one) unknown has to be assigned some convenient value so that all the unknowns can be determined by solving the n equations". This still remain an issue since there is no lay down procedure for assigning this so-called convenient value to a particular unknown in the system of the equations. Furthermore, it is some worth ironical to observe that the auxiliary function constructed by Kulkarni is not applicable to $\sum_{j=0}^{n} a_{n-j} x^{n-j} = 0; a_n \neq 0; if\ n = 1$ which contradict the claim that it is valid for $n < 5$, since $n = 1 < 5$ and this is easily seen from the equation that determine the number of unknown which is given by

$$2m + 1 = \begin{cases} n & ; for\ n\ odd \\ n + 1 & ; for\ n\ even \end{cases}$$

This has no solution for $n = 1$, since this implies that $m = 0$, suggesting that the auxiliary function in equation (1.1) is a constant. This cannot be!, thus a contradiction. Furthermore, observe that for every $k > 1$, $km = n\ and\ n = 1$ implies that $1 = n = km = k(0) = 0 \Rightarrow\Leftarrow$ again, a contradiction.

In order to do away with the above anormally, an alternative approach to the auxiliary function (polynomial) constructed by Kalkarni shall be introduced, and in addition, enjoys the following properties (features) over the one constructed by Kulkarni;

(i) It is simpler than the one constructed in equation (1.1).

(ii) The degree of $V_m(x)$ is not necessarily equal to the degree of $W_m(x)$. That is $dg(V_m(x)) = dg(V_m(x)) = m$ is dispensed with.

(iii) The issue of assigning some convenient value to an extra (one) unknown in arbitrary manner is also dispensed with.

(iv) An auxiliary function that still remains valid even for a linear equation, that is when $n = 1$ is presented.

2. PRELIMINARY

In this section we shall define some important terms as will be needed in the presentation of this work. Let $\mathbb{N}$ be the set of natural numbers and D the set of real (or complex) numbers. Then for any given $n \in \mathbb{N}$, a function $f_n: D \longrightarrow D$ is said to be a polynomial of degree n if there exists a constant $a_j (j = 0, 1, 2, \ldots, n)$ such that f is given by

$$f_n(x) = \sum_{j=0}^{n} a_{n-j} x^{n-j} \,; a_n \neq 0 - - - - (1.2)$$

Equation (1.2) is said to be Depressed if $a_{n-1} = 0$, Monic if $a_n = 1$. And we define the set of first n positive integers by

$$[n] := \{1, 2, 3, \cdots, n\} - - - - (1.3)$$

At this juncture, we are now ready to define our proposed auxiliary function which we shall give in the theorem that follows and then give a formal proof to the fact that our auxiliary function is indeed faithful for every $n \leq 4$.

Theorem 2.1

Let $n, k \in \mathbb{N}$ and $f_n(x) = 0$ a polynomial equation of degree n $(n \leq 4)$, then there exist u, v, w an undetermined parameters such that the zero solution to the auxiliary function

$$g(y; u, v, w) = (vy + w)^k - (y^{r_n} + u)^k \text{: for some } r_n \in [n] - - - (1.4a)$$

$$x = y - \frac{b}{na} - - - - (1.4b)$$

$$kr_n = n \; - - - - - (1.4c)$$

solves the given polynomial equation $f_n(x) = 0$.

3 THE LINEAR CASE

It is clear that when $n = 1$, then $f_1(x) = 0$ implies that $a_1 x + a_0 = 0$. We may simply write this as $ax + b = 0$

Now, for $n = 1$ there is at least one $r_1 \in [1] = \{1\}$ such that $kr_1 = 1$. Since $\{1\}$ is a singleton set, it suffices to take $r_1 = 1$, which implies that by equation (1.4c), $k = 1$. Thus by equation (1.4a) we have

$$g(y; u, v, w) = (vy + w) - (y + u)$$

Observe that $g(y; u, v, w) = 0;$

$$\Rightarrow (vy + w) - (y + u) = 0$$

$$\Rightarrow \ (v-1)y + (w-u) = 0 ----(1.5)$$

$$\Rightarrow \ (v-1)y = -(w-u);$$

$$\Rightarrow \ y = \frac{-(w-u)}{v-1} ----(1.6)$$

Is the seeking solution to the zero of the auxiliary function $g(.)$.

The depressed equation of equation (1.5) is

$$y = 0 \ ----(1.7)$$

To determine the unknown parameters, for completeness of procedure, we compare eq(1.5) and eq(1.7), we have

$$v - 1 = 1$$

$$w - u = 0 \ \Rightarrow u = w$$

Substituting this values into equation (1.7) we have

$$y = \frac{u-w}{v-1} = 0$$

So that by equation (1.4b) we have that

$$x = -\frac{b}{a}$$

Which is a general solution to the linear equation.

4 THE QUADRATIC CASE

It is clear that when $n = 2$, then $f_2(x) = 0$ implies that $a_2x^2 + a_1x + a_0 = 0$. We may simply write this as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \ \ (for\ a \neq 0) ----(1.8)$$

Now, for $n = 2$ there is at least one $r_2 \in [2] = \{1,2\}$ such that $kr_2 = 2$. It suffices to take $r_2 = 2$ which implies that $k = 1$. Thus by equation (1.4) we have

$$g(y; u, v, w) = (vy + w) - (y^2 + u)$$

Observe that $g(y; u, v, w) = 0;$

$$\Longrightarrow\ y^2 - vy + u - w = 0 - - - - (1.9)$$

$$\Longrightarrow\ y = \frac{v \pm \sqrt{v^2 - 4(u-w)}}{2} - - - (1.10)$$

This is what we call the seeking solution for the depressed quadratic equation.

Now, to obtain this depressed quadratic equation we substitute equation (1.4b) into equation (1.8) and obtain

$$y^2 + p = 0\ - - - (1.11)$$

Where $p = -\left(\frac{b^2-4ac}{4a^2}\right)$

Now to determine the unknown parameters u and w comparing equation (1.9) and equation (1.10), it follows that

$$v = 0\ and\ u - w = p\ - - - (1.12)$$

Substituting this values into equation (1.10) gives

$$y = \pm\sqrt{p} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

So that by equation (1.4b) we have that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Is a general solution to the quadratic equation in equation (1.8).

4 THE CUBIC CASE

It is clear that when $n = 3$, then $f_3(x) = 0$ implies that $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$. We may simply write this as

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0\ \ (for\ a \neq 0) - - - - (1.13)$$

Now, since $n = 3$ there is at least one $r_3 \in [3] = \{1,2,3\}$ such that $kr_3 = 3$. It suffices to take $r_3 = 1$ which implies that $k = 3$. Thus by equation (1.4a) we have

$$g(y; u, v, w) = (vy + w)^3 - (y + u)^3$$

And

$$g(y; u, v, w) = 0;$$

$$\Longrightarrow \quad (vy + w)^3 - (y + u)^3 = 0 - - - - (1.14)$$

$$\Longrightarrow \ (vy + w)^3 = (y + u)^3; \Longrightarrow \ vy + w = y + u; \Longrightarrow$$

$$y = \frac{u - w}{v - 1} - - - - (1.15)$$

Is the seeking solution to equation (1.6). Now using equation (1.4b) we obtain

$$y^3 + py + q = 0 \ - - - - (1.16)$$

a depressed cubic equation of (1.13). where $p = \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)$ and $q = \left(\frac{d}{a} + \frac{2b^3}{27a^3} - \frac{bc}{3a^2}\right)$.

In order to solve equation (1.13) we expand equation (1.14) and obtain

$$(v^3 - 1)y^3 + 3(v^2 w - u)y^2 + 3(vw^2 \ - u^2)y + w^3 - u^3 = 0 - - - - (1.17)$$

Now, comparing the coefficients in the two equations (1.6) and equation (1.13) we obtained

$$v^3 - 1 = 1 - - - (1.18a)$$

$$v^2 w - u = 0 - - - (1.18b)$$

$$vw^2 - u^2 = \frac{p}{3} - - - (1.18c)$$

$$w^3 - u^3 = q - - - (1.18d)$$

In equation (1.18b), it follows that $u = v^2 w$. Substituting into $(1.18c)$, $(1.18d)$ and using $(1.18a)$ we have

$$vw^2 = -\frac{p}{3} - - - (1.19)$$

$$(v^3 + 1)w^3 = -q - - - (1.20)$$

From equation (1.18c) $= -\frac{p}{3w^2}$, substituting this into (1.18d) and taking $t = w^3$ we have

$$27t^2 + 27qt - p^3 = 0 - - - (1.21)$$

is a quadratic equation in t whose solution is

$$t_j = -\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\ ; j = 1{,}2 - - - (1.22)$$

It follows that

$$w_j = \sqrt[3]{-\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\,; j = 1,2 - - - (1.23)$$

From (1.15), (1.18a) and (1.18b), we have $y = \frac{u-w}{v-1}, v = -\frac{p}{3w^2}$ and $u = v^2 w$ which implies $y = \left(w - \frac{p}{3}w^{-1}\right)$. Thus altogether with equation (1.32) we have

$$y_{1j} = \left(-\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{\frac{1}{3}} - \frac{p}{3}\left(-\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{-\frac{1}{3}}; j = 1,2 - - - (1.24)$$

Is a solution to (1.16). So that by equation (1.5) it follows that for each y_{1j} $(j = 1,2)$

$$x_{1j} = \left(-\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{\frac{1}{3}} - \frac{p}{3}\left(-\left(\frac{q}{2}\right) + (-1)^j \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{-\frac{1}{3}} - \frac{b}{3a}; j = 1,2 - - - (1.25)$$

Is a solution to (1.13). However for each j $(j = 1,2)$, x_{1j} is one of the three roots and the remaining roots x_{2j}, x_{3j} can be found by employing Vieta identity formula for relationship between roots given by:

$$y_{1j} + y_{2j} + y_{3j} = 0,\ y_{1j}y_{2j} + y_{2j}y_{3j} + y_{3j}y_{1j} = p$$

From these we get for every fixed j a quadratic equation whose solution gives

$$y_{ij} = \left(\frac{-y_{1j}}{2}\right) + (-1)^i \sqrt{-\left(\frac{3}{4}\right) y_{1j}{}^2 - p}; i = 2,3 - - - -(1.26)$$

The above relations and (1.4b) completely define the solution of the general cubic equation. Hence, for each fixed j $(j = 1,2)$, we obtain x_{ij} $(i = 1,2,3)$. Thus we have shown that $\forall\ j\ \in \{1, 2\}$, there exist y_{ij} (solution to (1.16)) such that

$$x_{ij} = y_{ij} - \frac{b}{3a}\ ; i = 1,2,3$$

Is a solution to equation (1.13).

4 THE QUARTIC CASE

It is clear that when $n = 4$, then $f_n(x) = 0$ implies that $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$. We may simply write this as

$$x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0\ \ (for\ a\neq 0)----(1.27)$$

Now, since $n=4$ there is at least one $r_4\in[4]=\{1,2,3,4\}$ such that $kr_4=4$. It suffices to take $r_4=2$ which implies that $k=2$. So that

$$g(y;u,v,w)=(vy+w)^2-(y^2+u)^2$$

Using equation (1.4b) we obtain

$$y^4+py^2+qy+r=0\ ----(1.28)$$

A depressed quartic equation of (1.27). where $p=\left(\frac{c}{a}-\frac{3b^2}{8a^2}\right)$, $q=\left(\frac{d}{a}-\frac{bc}{2a^2}+\frac{b^3}{8a^3}\right)$ and

$r=\left(\frac{e}{a}-\frac{bd}{4a^2}+\frac{b^2c}{16a^3}-\frac{3b^4}{256a^4}\right)$

In order to solve equation (1.28), first we consider the solution to

$$g(y;u,v,w)=0;$$

$$\Longrightarrow\ (vy+w)^2-(y^2+u)^2=0----(1.29)$$

$$\Longrightarrow\ (y^2+u)^2=(vy+w)^2;\Longrightarrow y^2+u=\pm(\,vy+w);$$

$$\Longrightarrow\ \ y^2+u=(\,vy+w)\text{ or }y^2+u=-(\,vy+w);$$

$$\Longrightarrow\ \ y^2-vy+(u-w)=0\ \text{ or }\ y^2+vy+(u+w)=0;$$

$$\Longrightarrow\ \ y_1=\frac{v-\sqrt{v^2-4(u-w)}}{2}\ or\ y_2=\frac{v+\sqrt{v^2-4(u-w)}}{2}\ ---(1.30a)$$

$$\Longrightarrow\ \ y_3=\frac{-v-\sqrt{v^2-4(u+w)}}{2}\ \ or\ \ y_4=\frac{-v+\sqrt{v^2-4(u+w)}}{2}\ ---(1.30b)$$

Constitute the seeking solution to the quartic equation (1.28). Now expanding (1.29) we obtain

$$y^4+(2u-v^2)y^2-2wvy+u^2-w^2=0----(1.31)$$

Now, comparing the coefficients in the two equations (1.28) and (1.31) we obtained

$$p=(2u-v^2)\Rightarrow u=\frac{p+v^2}{2}----(1.32)$$

$$q=-2wv\Rightarrow w=\frac{-q}{2v}----(1.33)$$

$$r = (u^2 - w^2); \Longrightarrow\ r = \left(\frac{p+v^2}{2}\right)^2 - \left(\frac{-q}{2v}\right)^2 ----(1.34)$$

We expand (1.34) to have

$$v^6 + 2pv^4 + (p^2 - 4r)v^2 - q^2 = 0\ ---\ (1.35)$$

Which is the resolvent equation associated with the quartic equation.

Let

$$v^2 = z\ ---\ (1.36)$$

Then by substituting into (1.35) we have

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0$$

Thus, using any standard method for solving cubic equation we obtain at least one solution in z, consequently v using the relating equation (1.36).

Now, recall from (1.32) and (1.33) that $u = \frac{p+v^2}{2}$ and $w = \frac{-q}{2v}$ which implies that

$$u + w = \frac{v(v^2+p) - q}{2v}\ and\ u - w = \frac{v(v^2+p) + q}{2v}$$

Then the above seeking solution to (1.28) becomes

$$y_1 = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}}\ \ or\ \ y_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}}$$

$$y_3 = -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}}\ \ or\ \ y_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}}$$

Consequently the solution to (1.27) becomes

$$x_1 = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} - \frac{b}{4a}\ \ or\ \ x_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} - \frac{b}{4a}$$

$$x_3 = -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} - \frac{b}{4a}\ \ or\ \ x_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} - \frac{b}{4a}$$

Alternatiely, this can be simply put as

$$x_{k,j} = \frac{(-1)^k v^2 + (-1)^j \sqrt{v^4 - 2v\left(v(v^2+p) + (-1)^k q\right)}}{2v} - \frac{b}{4a}; k = 1, 2;\ j = 1,2$$

5. APPLICATIONS

It now remains to apply and demonstration with telling examples the validity of our results for solving $f_n(x) = 0$ for every $n \in N$ such that $n < 5$. Clearly for $n = 1$ and $n = 2$ is quite trivial, hence the examples that follows justifies our results for the cases $n = 3$ and $n = 4$.

Example 1: Find the value of x given that: $x^3 - x^2 - 5x - 3 = 0$

Solution

$$y_{1j} = \left(-\left(\frac{q}{2}\right) + (-1)^j\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{\frac{1}{3}} - \frac{p}{3}\left(-\left(\frac{q}{2}\right) + (-1)^j\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{-\frac{1}{3}}$$

Where $a = 1,\ b = -1,\ c = -5,\ d = -3$

Then,

$$p = \frac{-b^2}{3a^2} + \frac{c}{a} = \frac{-(-1)^2}{3\times(1)^2} + \frac{-5}{1} = \frac{-16}{3}$$

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = \frac{2\times(-1)^3}{27\times(1)^3} - \frac{(-1)\times(-5)}{3\times(1)^2} + \frac{(-3)}{1} = \frac{-128}{27}$$

$$y_{1j} = \left(\frac{-\left(\frac{-128}{27}\right)}{2} + (-1)^j\sqrt{\frac{\left(\frac{-128}{27}\right)^2}{4} + \frac{\left(\frac{-16}{3}\right)^3}{27}}\right)^{\frac{1}{3}} - \frac{\left(\frac{-16}{3}\right)}{3}\left(\frac{-\left(\frac{-128}{27}\right)}{2} + (-1)^j\sqrt{\frac{\left(\frac{-128}{27}\right)^2}{4} + \frac{\left(\frac{-16}{3}\right)^3}{27}}\right)^{-\frac{1}{3}}$$

$$= \left(\frac{128}{54} + (-1)^j\sqrt{\frac{16384}{2916} - \frac{4096}{729}}\right)^{\frac{1}{3}} + \left(\frac{16}{9}\right)\left(\frac{128}{54} + (-1)^j\sqrt{\frac{16384}{2916} - \frac{4096}{729}}\right)^{-\frac{1}{3}}$$

$$= \left(\frac{128}{54} + (-1)^j\sqrt{\frac{16384-16384}{2916}}\right)^{\frac{1}{3}} + \left(\frac{16}{9}\right)\left(\frac{128}{54} + (-1)^j\sqrt{\frac{16384-16384}{2916}}\right)^{-\frac{1}{3}} = \frac{8}{3}\ (twice)$$

Thus,

$$x_{1j} = y_{1j} - \frac{b}{3a} = \frac{8}{3} - \frac{-1}{3\times1} = 3$$

the remaining roots x_{2j}, x_{3j} can be found by employing

$$y_{kj} = \left(\frac{-y_{1j}}{2}\right) + (-1)^k\sqrt{-\left(\frac{3}{4}\right)y_{1j}{}^2 - p}\,;\quad k = 2,3$$

$$= \left(\frac{-\frac{8}{3}}{2}\right) + (-1)^k\sqrt{-\left(\frac{3}{4}\right)\left(\frac{8}{3}\right)^2 - \frac{-16}{3}} = \left(\frac{-\frac{8}{3}}{2}\right) + (-1)^k\sqrt{-\frac{16}{3} + \frac{16}{3}} = -\frac{4}{3}\ (twice)$$

Hence

$$x_{kj} = y_{kj} - \frac{b}{3a} = -\frac{4}{3} - \frac{-1}{3 \times 1} = -1\ (twice)$$

$$\Longrightarrow\ \forall\ j = 1,2\ ;\ x_{kj} = 3, -1, -1\ \ ; k = 1,2,3 \text{ respectively.}$$

Example 2: Find the roots of

$$x^4 + 4x^3 + 33x^2 + (58 - 14i)x + (148 - 14i) = 0$$

Solution

Reduce above equation into depressed quartic equation by substituting $x = y - \frac{b}{4a} \Rightarrow x = y - \frac{4}{4\times 1} \Rightarrow x = y - 1$ to obtain

$$y^4 + 27y^2 - 14iy + 120 = 0$$

Using resolvent cubic equation; that is

$$z^3 + 2z^2p + (p^2 - 4r)z - q^2 = 0$$

where $p = 27, q = -14i,\ r = 120$ substitute this values we have

$$z^3 + 54z^2 + 249z + 196 = 0$$

We can use any standard method to obtain the first root of the resolvent cubic equation, hence we see that $z = -1$ is one of such root. Since $v^2 = z \Rightarrow v = \pm i$.

Using $v = i,\ a = 1\ , b = 4,\ p = 27,\ q = -14i$ we see that the roots of equation (82) are

$$x_1 = \frac{-b}{4a} + \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} = \frac{-4}{4} + \frac{i}{2} + \sqrt{\frac{i^2}{4} - \frac{i(i^2+27)+(-14i)}{2i}} = -1 + 3i$$

$$x_2 = \frac{-b}{4a} + \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} = \frac{-4}{4} + \frac{i}{2} - \sqrt{\frac{i^2}{4} - \frac{i(i^2+27)+(-14i)}{2i}} = -1 - 2i$$

$$x_3 = \frac{-b}{4a} - \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} = \frac{-4}{4} - \frac{i}{2} + \sqrt{\frac{i^2}{4} - \frac{i(i^2+27)-(-14i)}{2i}} = -1 + 4i$$

$$x_4 = \frac{-b}{4a} - \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} = \frac{-4}{4} - \frac{i}{2} - \sqrt{\frac{i^2}{4} - \frac{i(i^2+27)-(-14i)}{2i}} = -1 - 5i$$

Note that if we use $v = -i$ we obtain the same values for the roots x_j $(j = 1,2,3,4)$ above.

$\Longrightarrow \;\; x_j = -1+3i, -1-2i, -1+4i, -1-5i \;\; ; j = 1,2,3,4$ respectively.

Example 3: Find the roots of $y^4 - y^2 + 2y + 2$

Solution

Since the equation is a depressed quartic equation we solve using resolvent cubic equation that is

$$z^3 + 2z^2p + (p^2 - 4r)z - q^2 = 0$$

Observe that $p = -1, q = 2, \; r = 2$, Substituting this values we have

$$z^3 - 2z^2 - 7z - 4 = 0$$

We can use any standard method to obtain the first root of the resolvent cubic equation, hence we see that $z = 4$ is one of such root. Since $v^2 = z \Rightarrow v = \pm 2$.

Using $v = 2, \; a = 0\,, b = 0, \; p = -1, \; q = 2,$ then we observe the roots are

$$x_1 = \frac{-b}{4a} + \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} = \frac{0}{4} + \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2-1)+2}{2\times 2}} = 1 + i$$

$$x_2 = \frac{-b}{4a} + \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)+q}{2v}} = \frac{0}{4} + \frac{2}{2} - \sqrt{\frac{2^2}{4} - \frac{2(2^2-1)+2}{2\times 2}} = 1 - i$$

$$x_3 = \frac{-b}{4a} - \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} = \frac{0}{4} - \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2-1)-2}{2\times 2}} = -1$$

$$x_4 = \frac{-b}{4a} - \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2+p)-q}{2v}} = \frac{0}{4} - \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2-1)-2}{2\times 2}} = -1$$

$\Longrightarrow \;\; x_j = 1+i, 1-i, -1, -1 \;\; ; j = 1,2,3,4$ respectively.

6. CONCLUSION

We have successfully constructed an alternative approach to the auxiliary function (polynomial) constructed by Kalkarni that enjoys the above features itemized in (i) to (iii) over the one constructed by Kulkarni (2006). In particular, over the one constructed by Kulkarni (2006), Kalman and White (2001) due to property (iv) above. Also, we have demonstrated with telling examples the validity and simplicity of our result and method when compared with the methods employed by some authors in the literature.

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