ON AN INEQUALITY IN L^q VERSION

Barchand Chanam

Department of Mathematics

National Institute of Technology Manipur,Langol, Imphal-795004, Manipur, India

Email:barchand_2004@yahoo.co.in

Abstract

If $f(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ is a polynomial of degree n which does not vanish in $|z|<t$, where $t\geq 1$, then for $0<r\leq \rho \leq t$, Dewan and Mir **[** Int. J. Math. Math. Scs., 16(2005), 2641-2645] proved

$$\max_{|z|=\rho}|f'(z)|\leq n\frac{(\rho+t)^{n-1}}{(t+r)^n}\left\{1-\frac{t(t-\rho)(n|c_0|-t|c_1|)n}{(t^2+\rho^2)n|c_0|+2t^2\rho|c_1|}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\max_{|z|=r}|f(z)|.$$

In this paper, we prove an interesting inequality, which not only extends the above result to L^q version as a particular case, but also gives some interesting known results as corollaries.

Keywords: *Polynomial, L^q Inequalities, Maximum Modulus.*

INTRODUCTION

Let $f(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ be a polynomial of degree n and $f'(z)$ be its derivative. We define

$$\|f\|_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}, 0<q<\infty \tag{1.1}$$

If we let $q\to\infty$ in the above equality and make use of of the well-known fact from analysis [17] that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}=\max_{|z|=1}|f(z)|,$$

we can suitably denote

$$\|f\|_\infty=\max_{|z|=1}|f(z)|.$$

Similarly, one can define $\|f\|_0 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(e^{i\theta}\right)\right|d\theta\right\}$ and show that $\lim_{q\to 0+}\|f\|_q = \|f\|_0$. It would be of further interest that by taking limits as $q \to 0+$, the stated result holding for $q>0$, holds for $q=0$ as well.

For $r>0$, we denote by $M(f,r)=\max_{|z|=r}|f(z)|$ and accordingly $\|f\|_\infty = \max_{|z|=1}|f(z)| = M(f,1)$.

A famous result due to Bernstein [13 or also see 18] states that if $f(z)$ is a polynomial of degree n, then

$$\|f'\|_\infty \le n\|f\|_\infty. \tag{1.2}$$

Inequality (1.2) can be obtained by letting $q\to\infty$ in the inequality

$$\|f'\|_q \le n\|f\|_q. \tag{1.3}$$

Inequality (1.3) for $q\ge 1$ is due to Zygmund [19]. Arestov [1] proved that (1.3) remains valid for $0<q<1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then inequalities (1.2) and (1.3) can be respectively improved by

$$\|f'\|_\infty \le \frac{n}{2}\|f\|_\infty. \tag{1.4}$$

and

$$\|f'\|_q \le \frac{n}{\|1+z\|_q}\|f\|_q, \; q>0. \tag{1.5}$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [11], whereas, inequality (1.5) was proved by de-Bruijn [6] for $q\ge 1$. Rahman and Schmeisser [14] showed that (1.5) remains true for $0<q<1$.

As a generalization of (1.4), Malik [12] proved that if $f(z)$ does not vanish in $|z|<t$, $t\ge 1$, then

$$\|f'\|_\infty \le \frac{n}{1+t}\|f\|_\infty. \tag{1.6}$$

Under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [8] extended inequality (1.6) to L^q norm by showing that

$$\|f'\|_q \leq \frac{n}{\|t+z\|_q}\|f\|_q \ , q \geq 1. \tag{1.7}$$

It was shown by Gardner and Weems [7] and independently by Rather [16] that (1.7) also holds for $0 < q < 1$.

Further, as a generalization of (1.6), Bidkham and Dewan [3] proved that if $f(z)$ is a polynomial of degree n having no zero in $|z| < t$, $t \geq 1$, then

$$\|f'(rz)\|_\infty \leq n\frac{(r+t)^{n-1}}{(1+t)^n}\|f\|_\infty \text{ for } 1 \leq r \leq t. \tag{1.8}$$

For the same class of polynomials $f(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$, by involving certain coefficients, Dewan and Mir [5] improved as well as generalized inequality (1.8) by proving

$$\|f'(\rho z)\|_\infty \leq n\frac{(\rho+t)^{n-1}}{(r+t)^n}\left\{1 - \frac{t(t-\rho)(n|c_0|-|c_1|t)n}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\|f(rz)\|_\infty \ . \tag{1.9}$$

for $0 \leq r \leq \rho \leq t$.

In this paper, we prove an inequality in L^q norm, which not only extends inequality (1.9) to L^q version as a particular case, but also gives some interesting known results as corollaries. More precisely, we prove

Theorem. *If* $f(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z| < t$, $t > 0$, then $0 < r \leq \rho \leq t$.

$$\|f'(\rho z)\|_q \leq nT_q\left\{\int_0^{2\pi}\left[\left|f\left(re^{i\theta}\right)\right| + (\rho+t)M(f,r) \times \frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n - 1\right\}\right]^q d\theta\right\}^{\frac{1}{q}} . \tag{1.10}$$

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

Remark 1.1. If we use the fact that $\left|f\left(re^{i\theta}\right)\right|\le M\left(f,r\right)=\left\|f\left(rz\right)\right\|_\infty$ for each $\theta\in\left[0,\ 2\pi\right)$ and Lemma 2.8, we obtain the following improved version of inequality (1.9) in L^q norm.

Corollary 1.1. *If* $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<t$, $t>0$,then $0<r\le\rho\le t$.

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}\le nT_q\left(\frac{\rho+t}{r+t}\right)^n\left[1-\left\{\frac{t\left(t-\rho\right)\left(n|c_0|-|c_1|t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\right]\left\|f\left(rz\right)\right\|_\infty. \tag{1.11}$$

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

Remark 1.2. If $0<r<\rho\le t$, we have

$$\begin{aligned}1-\left(\frac{t+r}{t+\rho}\right)^n&=\frac{\left(\rho-r\right)}{\left(\rho+t\right)\left\{1-\left(\frac{r+t}{\rho+t}\right)\right\}}\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\\&=\left(\frac{\rho-r}{t+\rho}\right)\left\{\left(\frac{t+r}{t+\rho}\right)^{n-1}+\left(\frac{t+r}{t+\rho}\right)^{n-2}+............+\left(\frac{t+r}{t+\rho}\right)+1\right\}\\&\ge\left(\frac{\rho-r}{t+\rho}\right)n\left(\frac{t+r}{t+\rho}\right)^{n-1}.\end{aligned} \tag{1.12}$$

Also, for $r=\rho$, inequality (1.12) holds trivially and hence inequality (1.12) is true for $0<r\le\rho\le t$. Using this fact in Corollary 1.1, we obtain the direct L^q analogue of (1.9) due to Dewan and Mir [7] with extended value of the radius t of the zero free open disc from $t\ge 1$ to $t>0$.

Corollary 1.2. *If* $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<t$, $t>0$, then $0<r\le\rho\le t$.

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}\le nT_q\left(\frac{\rho+t}{r+t}\right)^n\left[1-\left\{\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)n}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\right]\left\|f(rz)\right\|_\infty$$

(1.13)

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

Remark 1.3. Taking limit as $q\to 0+$ on both side of (1.13), we obtain inequality (1.9) as

Corollary 1.3. *If* $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<t$, $t>0$, then $0<r\le\rho\le t$.

$$\left\|f'(\rho z)\right\|_\infty\le n\frac{(\rho+t)^{n-1}}{(r+t)^n}\left\{1-\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)n}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\left\|f(rz)\right\|_\infty. \qquad (1.14)$$

Remark 1.4. If we set $r=1$ and $\rho=r$ in corollary 1.3, we have an improvement of (1.8). Further, putting $1=r=\rho$ both the corollaries 1.1 and 1.2 reduce to inequality (1.7).

2. LEMMAS

For the proof of the theorem the following lemmas are required.

Lemma 2.1. *If* $f(z)=\sum_{\upsilon=1}^{n}c_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z|<t$, $t\ge 1$, *then*

$$\max_{|z|=1}\left|f'(z)\right|\le\frac{n}{1+t}\max_{|z|=1}\left|f(z)\right|. \qquad (2.1)$$

This result is due to Malik [12].

Lemma 2.2. *If* $f(z)=\sum_{\upsilon=1}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then for* $|z|=1$,

$$t|f(z)|\leq |g'(z)|. \tag{2.2}$$

where $g(z)=z^{n}\overline{f\left(\frac{1}{\bar{z}}\right)}$.

Malik [12, Lemma 3] proved this lemma.

Lemma 2.3. *If* $f(z)=\sum_{\upsilon=1}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then*

$$\max_{|z|=1}|f'(z)|\leq n\frac{n|c_0|+t^2|c_1|}{(1+t^2)n|c_0|+2t^2|c_1|}\max_{|z|=1}|f(z)|. \tag{2.3}$$

This result was proved by Govil at. el. [9].

Lemma 2.4. *If* $f(z)=c_0+\sum_{\upsilon=\mu}^{n} c_{\upsilon} z^{\upsilon}, 1\leq \mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then*

$$\frac{\mu}{n}\left|\frac{c_{\mu}}{c_0}\right|t^{\mu}\leq 1. \tag{2.4}$$

Lemma 2.3 is due to Qazi [14, Remark 1].

Lemma 2.5. *If* $f(z)=c_0+\sum_{\upsilon=\mu}^{n} c_{\upsilon} z^{\upsilon}, 1\leq \mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then the function*

$$p(x)=\frac{\left(n|c_0|x+|c_1|t^2\right)(t+x)}{\left(x^2+t^2\right)n|c_0|+2|c_1|t^2x}\ , \tag{2.5}$$

is a non-decreasing function of t in $(0,t]$.

Proof of Lemma 2.5. We prove this by derivative test. Now, we have

$$p'(x)=\frac{\left(n|c_0|-|c_1|t\right)}{\left\{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x\right\}^2}\left\{(t-x)t\left(n|c_0|x+|c_1|t^2\right)+\left(n|c_0|+|c_1|t\right)\left(t^2x+t^3\right)\right\}$$

which is non-negative, since by Lemma 2.4, for $\mu=1$, $\left(n|c_0|-|c_1|t\right)\geq 0$, and the fact that $x\leq t$.

Lemma 2.6. *If* $f(z)=\sum_{\upsilon=0}^{n}c_{\upsilon}z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$ *then for* $0<r\leq\rho\leq t$,

$$\max_{|z|=r}\left|f(z)\right|\geq\left(\frac{r^{\mu}+t^{\mu}}{\rho^{\mu}+t^{\mu}}\right)^{\frac{n}{\mu}}\max_{|z|=\rho}\left|f(z)\right|. \tag{2.6}$$

This lemma was proved by Jain [10].

Lemma 2.7. *If* $f(z)=\sum_{\upsilon=0}^{n}c_{\upsilon}z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then for* $0<r\leq\rho\leq t$,

$$\left|f\left(\rho e^{i\theta}\right)\right|\leq\left|f\left(re^{i\theta}\right)\right|+(\rho+t)M(f,r)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}.$$

Proof of Lemma 2.7. Since $f(z)$ does no vanish in $|z|<t$, $t>0$, the polynomial $F(z)=f(xz)$ where $0<x\leq t$ has no zero in $|z|<\frac{t}{x}$, where $\frac{t}{x}\geq 1$. Hence applying Lemma 2.3 to the polynomial $F(z)$, we get

$$\max_{|z|=1}\left|F'(z)\right|\leq n\left[\frac{n|c_0|+|c_1x|\left(\frac{t}{x}\right)^2}{\left\{1+\left(\frac{t}{x}\right)^2\right\}n|c_0|+2|c_1x|\left(\frac{t}{x}\right)^2}\right]\max_{|z|=1}\left|F(z)\right|,$$

which implies

$$\max_{|z|=t}\left|f'(z)\right|\leq n\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x}\right\}\max_{|z|=t}\left|f(z)\right|. \tag{2.7}$$

Now, for $0<r\leq\rho\leq t$ and $0\leq\theta<2\pi$, we have on using (2.7)

$$\begin{aligned}\left|f\left(\rho e^{i\theta}\right)-f\left(re^{i\theta}\right)\right|&\leq\int_r^{\rho}\left|f'\left(xe^{i\theta}\right)\right|dx\\&\leq\int_r^{\rho}n\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x}\right\}\max_{|z|=t}\left|f(z)\right|dx,\end{aligned}$$

Appling Lemma 2.6, we have

$$\left|f\left(\rho e^{i\theta}\right)-f\left(re^{i\theta}\right)\right|\le\int_r^{\rho}n\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x}\right\}\times\left(\frac{t+x}{t+r}\right)^n M\left(f,r\right)dx$$

$$=\frac{nM\left(f,r\right)}{\left(r+t\right)^n}\int_r^{\rho}\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|t}\right\}\left(x+t\right)^n dx. \tag{2.8}$$

For $0<r\le x\le\rho\le t$, by Lemma 2.5, we have

$$\frac{\left(n|c_0|x+|c_1|t^2\right)\left(x+t\right)}{\left(x^2+t^2\right)n|c_0|+2x|c_1|t^2}\le\frac{\left(n|c_0|\rho+|c_1|t^2\right)\left(\rho+t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}. \tag{2.9}$$

Combining (2.8) with (2.9), we get

$$\left|f\left(\rho e^{i\theta}\right)-f\left(re^{i\theta}\right)\right|\le\frac{n\left(\rho+t\right)M\left(f,r\right)}{\left(r+t\right)^n}\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\int_r^{\rho}\left(x+t\right)^{n-1}dx$$

$$=\left(\rho+t\right)M\left(f,r\right)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\},$$

from which it is implied by triangle inequality that

$$\left|f\left(\rho e^{i\theta}\right)\right|\le\left|f\left(re^{i\theta}\right)\right|+\left(\rho+t\right)M\left(f,r\right)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\},$$

which completes the proof of Lemma 2.7.

Lemma 2.8. *If* $f\left(z\right)=\sum_{\upsilon=0}^{n}c_{\upsilon}z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then for* $0<r\le\rho\le t$,

$$(\rho+t)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}=\left(\frac{\rho+t}{r+t}\right)^n \times\left[1-\left\{\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\right]-1$$

Proof of Lemma 2.8. We have

$$(\rho+t)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}=\left(\frac{\rho+t}{r+t}\right)^n \times\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\left\{\frac{(\rho+t)\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\} \tag{2.10}$$

Now

$$\left\{\frac{(\rho+t)\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}=1-\left\{\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}. \tag{2.11}$$

Using (2.11) in (2.10), we get

Lemma 2.9. *If* $f(z)$ *is a polynomial of degree* n *and* $g(z)=z^n\overline{f\left(\frac{1}{\bar{z}}\right)}$, *then for each* α, $0\le\alpha<2\pi$ *and* $r>0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|g'\left(e^{i\theta}\right)+e^{i\alpha}f'\left(e^{i\theta}\right)\right|^r d\theta\, d\alpha\le 2\pi n^r\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^r d\theta\,. \tag{2.10}$$

The above lemma is due to Aziz and Rather [2].

Lemma 2.10. *Let z be complex and independent of* α, *where* α *is real, then for* $p>0$,

$$\int_0^{2\pi}\left|1+ze^{i\alpha}\right|^p d\alpha=\int_0^{2\pi}\left|e^{i\alpha}+|z|\right|^p d\alpha\,. \tag{2.11}$$

This lemma is due to Govil [6].

2. PROOF OF THE THEOREM

Proof of the Theorem. Since the polynomial $f(z)$ has no zero in $|z|<t$, $t>0$, the polynomial $F(z)=f(\rho z)$ has no zero in $|z|<\frac{t}{\rho}, \frac{t}{\rho}\geq 1$.By applying Lemma 2.2 to $F(z)$, we have for $|z|=1$,

$$\frac{t}{\rho}|F'(z)|\leq|G'(z)| \text{ for} |z|=1, \tag{3.1}$$

where $G(z)=z^n\overline{F\left(\frac{1}{\overline{z}}\right)}$.

We can easily verify that for every real number α and $R\geq r'\geq 1$,

$$\left|R+e^{i\alpha}\right|\geq\left|r'+e^{i\alpha}\right|.$$

This implies for each $q>0$,

$$\int_0^{2\pi}\left|R+e^{i\alpha}\right|^q d\alpha\geq\int_0^{2\pi}\left|r'+e^{i\alpha}\right|^q d\alpha. \tag{3.2}$$

For points $e^{i\theta}, 0\leq\theta<2\pi$, for which $P'(e^{i\theta})\neq 0$,we denote

$$R=\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|,$$

and $r'=\frac{k}{\rho}$ then from (3.1),

$$R\geq r'\geq 1.$$

Now, we have for each $q>0$,

$$\int_0^{2\pi}\left|G'\left(e^{i\theta}\right)+e^{i\alpha}F'\left(e^{i\theta}\right)\right|^q d\alpha=\left|F'\left(e^{i\theta}\right)\right|^q\int_0^{2\pi}\left|\frac{G'\left(e^{i\theta}\right)}{F'\left(e^{i\theta}\right)}+e^{i\alpha}\right|^q d\alpha.$$

$$=\left|F'\left(e^{i\theta}\right)\right|^q\int_0^{2\pi}\left|\left|\frac{G'\left(e^{i\theta}\right)}{F'\left(e^{i\theta}\right)}\right|+e^{i\alpha}\right|^q d\alpha \text{ (by Lemma 2.10)}$$

$$\geq\left|F'\left(e^{i\theta}\right)\right|^q\int_0^{2\pi}\left|\frac{t}{\rho}+e^{i\alpha}\right|^q d\alpha \text{ ,by(3.2)} \tag{3.3}$$

For points $e^{i\theta}$, $0 \le \theta < 2\pi$, for which $F'\left(e^{i\theta}\right)=0$, inequality (3.3) trivially holds.

Now using (3.3) in Lemma 2.9, we obtain for each $q>0$,

$$\int_0^{2\pi}\left|\frac{t}{\rho}+e^{i\alpha}\right|^q d\alpha \int_0^{2\pi}\left|F'\left(e^{i\theta}\right)\right|^q d\theta \le 2\pi n^q \int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^q d\theta,$$

which is equivalent to

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|F'\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le nS_q\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}},$$

where

$$S_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{t}{\rho}+e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

Since $F(z)=f(\rho z)$, $F'(z)=\rho f'(\rho z)$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le \frac{n}{\rho}S_q\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}},$$

This in conjunction with lemma 2.7 and noting $\frac{S_q}{\rho}=T_q$, we obtain

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le nT_q\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\left|f\left(re^{i\theta}\right)\right|+(\rho+t)M(f,r)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\times\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}\right]^q d\theta\right\}^{\frac{1}{q}}$$

This completes the proof of the Theorem.

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