SUMMATION FORMULAE INVOLVING BASIC HYPERGEOMETRIC AND TRUNCATED BASIC HYPERGEOMETRIC FUNCTIONS

[1]Harwinder Kaur & [2]A.K.Shrivastav

[1]*Department of Mathematics*
RIMT University, Mandi Gobindgarh ,Fathegarh Sahib, Punjab (India).
Email: harwinderkaur808@gmail.com

[2]*Department of Mathematics*
RIMT University, Mandi Gobindgarh, Fathegarh Sahib, Punjab (India).
Email: aakkaasshhkumar8888@gmail.com

Abstract

In this paper, making use of concept splitting the limits of summation identities and Bailey's lemma, an attempt has been to establish the transformations involving Bailey's lemma and Truncated Basic Hypergeometric Series.

Keywords: *Bailey's lemma, q-series, Basic Hypergeometric functions, Truncated Basic Hypergeometric functions.*

INTRODUCTION

Bailey [2] established a simple but very useful identity

If

(1.1) $\beta_n = \sum_{r=0}^{n} u_{n-r} v_{n+r} \alpha_r$

and

(1.2)$\gamma_n = \sum_{r=n}^{\infty} u_{r-n} v_{r+n} \delta_r$

Where α_r, δ_r, u_r and v_r are any functions of r only such that, the series γ_n exists, then subject to the convergence of the series.

(1.3) $\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$

Making use of (1.3), a number of mathematicians , notably Slater[7] ,Verma [10] ,Verma and Jain[12], Denis[3], Gasper[4], Andrews[1], Shrivastav [6] others made use of Bailey's Lemma and gave a number of transformation formulae involving Basic Hypergeomtric Functions.In this paper, using concept of splitting the limits of summation identities, an attempt has been made to establish very interesting transformation formulae involving Truncated Basic Hypergeometric Functions.

DEFINITIONS AND NOTATIONS

The Basic Hypergeometric Functions is defined as;

(2.1) $\ {}_A\Phi_B \left[\begin{matrix} a_1, a_2, \ldots\ldots\ldots, a_A \ ; q; z \\ b_1, b_2, \ldots\ldots\ldots, b_B \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots\ldots (a_A;q)_n \, z^n}{(b_1;q)_n (b_2;q)_n \ldots\ldots (b_B;q)_n (q;q)_n}$

and

$(a_1, a_2, \ldots\ldots\ldots a_A; q)_n = (a_1; q)_n (a_2; q)_n \ldots\ldots\ldots (a_A; q)_n$

With the q-shifted factorial defined by

(2.2) $(a; q)_n = \begin{cases} 1, & if\ n=0 \\ (1-a)(1-aq)(1-aq^2)\ldots\ldots(1-aq^{n-1}), & if\ \ n=1,2,\ldots \end{cases}$

The Truncated Basic Hypergeometric Functions is defined as;

(2.3) $\ {}_A\Phi_B \left[\begin{matrix} a_1, a_2, \ldots\ldots\ldots\ldots, a_A \ ; q, z \\ b_1, b_2, \ldots\ldots\ldots . b_B \end{matrix} \right]_N = \sum_{n=0}^{N} \frac{(a_1;q)_n (a_2;q)_n \ldots\ldots (a_A;q)_n z^n}{(b_1;q)_n (b_2;q)_n \ldots\ldots (b_B;q)_n (q;q)_n}$

Where max $(|q|, |z| < 1)$ and no zero appears in the denominators.

Respectively, we shall use the following known identities in our script,

(2.4) $\quad (a; q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

(2.5) $\quad (a; q)_{n+k} = (a; q)_n (aq^n; q)_k$

(2.6) $\quad (aq^n; q)_k = \frac{(a;q)_k (aq^k;q)_n}{(a;q)_n}$

(2.7) $\quad (a; q)_{kn} = (a, aq, \ldots\ldots, aq^{k-1}; q^k)_n$

(2.8) $\quad (a^2; q^2)_n = (a, -a; q)_n$

(2.9) $\quad (a^3; q^3)_n = (a, a\omega, a\omega^2; q)_n$ where $\omega = e^{2\pi i/3}$

Also, we shall use the following well known q-series in our present script,

(2.10) $\ {}_0\Phi_0 \left[\begin{matrix} - \\ - \end{matrix} ; q; z \right] = (z; q)_\infty$ (Slater 8 [App. IV (IV.10)])

(2.11) $\ {}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix} ; q; z \right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, |q| < 1$ (Slater 8[App. IV (IV.10)])

(2.12) $\ {}_2\Phi_1 \left[\begin{matrix} a, b; q; c/ab \\ c \end{matrix} \right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}$ (Slater 8[App. IV (IV.10)])

(2.13) $\ {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-n}; q; q \\ c\ ,\ d \end{matrix} \right] = \frac{(c/a\ ; q)_n (c/b; q)_n}{(c\ ; q)_n (c/ab; q)_n}$ (Slater 8[App. IV (IV.4)])

(2.14) $${}_4\Phi_3\left[\begin{matrix} q^{-n}, bx^2q^{n+2}, x, -xq; q; q \\ xq\sqrt{b}, -xq\sqrt{b}, x^2q^2 \end{matrix}\right] = \frac{x^n[q;q]_n[bxq^2;q]_n[bx^2q^3;q^2]_m[bq^2;q^2]_m[xq^2;q]_{2m}}{[xq;q]_n[bx^2q^2;q]_n[q^2;q^2]_m[x^2q^3;q^2]_m[bxq^2;q]_{2m}}$$

(Verma and Jain [12 ;(3.2)])

(2.15) $${}_5\Phi_4\left[\begin{matrix} x, \omega xq, \omega^2 xq, x^3q^{n+4}, q^{-n} \quad ; q; q \\ (xq)^{\frac{3}{2}}, -(xq)^{\frac{3}{2}}, x^{\frac{3}{2}}q^2, -x^{\frac{3}{2}}q^2 \end{matrix}\right] = \frac{x^n[x^2q^4;q]_n[q;q]_n[x^3q^6;q^3]_n[xq^3;q]_{3m}}{[x^3q^4;q]_n[xq;q]_n[q^3;q^3]_m[x^2q^4;q]_{3m}}$$

(Verma and Jain [12])

(2.16) $${}_6\Phi_5\left[\begin{matrix} a^{\frac{1}{3}}, \omega a^{\frac{1}{3}}, \omega^2 a^{\frac{1}{3}}, q\sqrt{a}, aq^{n+1}, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{matrix}\right] = \frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(\sqrt{a})^{n-m}}{[aq;q]_n[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m}$$

(Verma and Jain [12])

(2.17) $${}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix} ; q; q\right]$$

$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}$$

(Jackson, F.H.,[5])

MAIN RESULTS

(3.1) $${}_0\Phi_0\left[\begin{matrix} - \\ - \end{matrix}; q; z\right]_n = z^{n+1}(z;q)_\infty + (1-z)\ {}_0\Phi_0\left[\begin{matrix} - \\ - \end{matrix}; q; z^2\right]$$

(3.2) $${}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q; z\right]_n = z^{n+1}\frac{(az;q)_\infty}{(z;q)_\infty} + (1-z)\ {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q; z^2\right]$$

(3.3) $${}_2\Phi_1\left[\begin{matrix} a, b; q; c/ab \\ c \end{matrix}\right]_n = z^{n+1}\frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty} + (1-z)\ {}_2\Phi_1\left[\begin{matrix} a, b; q; cz/ab \\ c \end{matrix}\right]$$

(3.4) $${}_3\Phi_2\left[\begin{matrix} a, b, q^{-n}; q; q \\ c, d \end{matrix}\right]_n = z^{n+1}\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} + (1-z)\ {}_3\Phi_2\left[\begin{matrix} a, b, q^{-n}; q; qz \\ c, d \end{matrix}\right]$$

(3.5) $${}_4\Phi_3\left[\begin{matrix} q^{-n}, bx^2q^{n+2}, x, -xq; q; q \\ xq\sqrt{b}, -xq\sqrt{b}, x^2q^2 \end{matrix}\right]_n = z^{n+1}\frac{x^n[q;q]_n[bxq^2;q]_n[bx^2q^3;q^2]_m[bq^2;q^2]_m[xq^2;q]_{2m}}{[xq;q]_n[bx^2q^2;q]_n[q^2;q^2]_m[x^2q^3;q^2]_m[bxq^2;q]_{2m}}$$

$$+ (1-z)\ {}_4\Phi_3\left[\begin{matrix} q^{-n}, bx^2q^{n+2}, x, -xq; q; zq \\ xq\sqrt{b}, -xq\sqrt{b}, x^2q^2 \end{matrix}\right]$$

(3.6) $${}_5\Phi_4\left[\begin{matrix} x,\ \omega xq,\ \omega^2 xq,\ x^3q^{n+4},\ q^{-n} \quad ; q, q \\ (xq)^{\frac{3}{2}}, -(xq)^{\frac{3}{2}}, x^{\frac{3}{2}}q^2, -x^{\frac{3}{2}}q^2 \end{matrix}\right]_n = z^{n+1}\frac{x^n[x^2q^4;q]_n[q;q]_n[x^3q^6;q^3]_n[xq^3;q]_{3m}}{[x^3q^4;q]_n[xq;q]_n[q^3;q^3]_m[x^2q^4;q]_{3m}}$$

$$+(1-z)\ {}_5\Phi_4\left[\begin{matrix} x\ ,\ \omega xq\ ,\ \omega^2 xq\ ,\ x^3q^{n+4},\ q^{-n}\ \ ;q\ ,q \\ (xq)^{\frac{3}{2}}, -(xq)^{\frac{3}{2}}, x^{\frac{3}{2}}q^2, -x^{\frac{3}{2}}q^2 \end{matrix}\right]$$

(3.7) $${}_6\Phi_5\left[\begin{matrix} a^{\frac{1}{3}}, \omega a^{\frac{1}{3}}, \omega^2 a^{\frac{1}{3}}, q\sqrt{a}\ , aq^{n+1}, q^{-n}; q, q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{matrix}\right]_n$$

$$= z^{n+1}\frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(\sqrt{a})^{n-m}}{[aq;q]_n[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m}$$

$$+(1-z){}_6\Phi_5\left[\begin{matrix} a^{\frac{1}{3}}, \omega a^{\frac{1}{3}}, \omega^2 a^{\frac{1}{3}}, q\sqrt{a}\ , aq^{n+1}, q^{-n}; q, qz \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{matrix}\right]$$

(3.8) $${}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}\ , b, c, d, a^2q^{1+n}/bcd\ , q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b\ , aq/c\ , aq/d\ , bcdq^{-n}/a\ , aq^{1+n} \end{matrix}\ ; q; q\right]_n$$

$$= z^{n+1}\ \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}$$

$$+(1-z)\ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}\ , b, c, d, a^2q^{1+n}/bcd\ , q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b\ , aq/c\ , aq/d\ , bcdq^{-n}/a\ , aq^{1+n} \end{matrix}\ ; q; zq\right]$$

PROOF OF MAIN RESULTS

In order to proof our main results, if we take $u_r = v_r = 1$, in (1.1) and (1.2).

We get

(4.1) $\beta_n = \sum_{r=0}^{n} \alpha_r$

and

(4.2) $\gamma_n = \sum_{r=n}^{\infty} \delta_r$

By substituting (4.1) and (4.2) in (1.3)

(4.3) $\sum_{n=0}^{\infty} \alpha_n (\sum_{r=n}^{\infty} \delta_r) = \sum_{n=0}^{\infty} (\sum_{r=0}^{n} \alpha_r)\delta_n$

By splitting the limits and simplifying, we get,

(4.5) $\sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=0}^{\infty} \alpha_n \delta_n$

Where $\alpha_n\ and\ \delta_n$ are arbitrary functions in the form of finite and infinite sequences .

Now, by taking $\alpha_r = \frac{(-;q)_r}{(q,-;q)_r}\ z^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.6) ${}_0\Phi_0\left[\begin{matrix}- \\ -\end{matrix};q;z\right]_n = z^{n+1}(z;q)_\infty + (1-z)\ {}_0\Phi_0\left[\begin{matrix}- \\ -\end{matrix};q;z^2\right]$

By taking $\alpha_r = \frac{(a;q)_r}{(q,-;q)_r} z^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.7) ${}_1\Phi_0\left[\begin{matrix}a \\ -\end{matrix};q;z\right]_n = z^{n+1}\frac{(az;q)_\infty}{(z;q)_\infty} + (1-z)\ {}_1\Phi_0\left[\begin{matrix}a \\ -\end{matrix};q;z^2\right]$

By taking $\alpha_r = \frac{(a,b;q)_r}{(q,c;q)_r}(c/ab)^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.8) ${}_2\Phi_1\left[\begin{matrix}a,b;q;c/ab \\ c\end{matrix}\right]_n = z^{n+1}\frac{[c/a,c/b;q]_\infty}{[c,c/ab;q]_\infty} + (1-z)\ {}_2\Phi_1\left[\begin{matrix}a,b;q;cz/ab \\ c\end{matrix}\right]$

By taking $\alpha_r = \frac{(a,b,q^{-n};q)_r\, q^r}{(q,c,d;q)_r}$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.9) ${}_3\Phi_2\left[\begin{matrix}a,b,q^{-n};q;q \\ c,d\end{matrix}\right]_n = z^{n+1}\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} + (1-z)\ {}_3\Phi_2\left[\begin{matrix}a,b,q^{-n};q;qz \\ c,d\end{matrix}\right]$

By taking $\alpha_r = \frac{(q^{-n},bx^2q^{n+2},x,-xq;q)_r}{(q,xq\sqrt{b},-xq\sqrt{b},x^2q^2;q)_r}(q)^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.10) ${}_4\Phi_3\left[\begin{matrix}q^{-n},bx^2q^{n+2},x,-xq;q;q \\ xq\sqrt{b},-xq\sqrt{b},x^2q^2\end{matrix}\right]_n = z^{n+1}\frac{x^n[q;q]_n[bxq^2;q]_n[bx^2q^3;q^2]_m[bq^2;q^2]_m[xq^2;q]_{2m}}{[xq;q]_n[bx^2q^2;q]_n[q^2;q^2]_m[x^2q^3;q^2]_m[bxq^2;q]_{2m}}$

$+ (1-z)\ {}_4\Phi_3\left[\begin{matrix}q^{-n},bx^2q^{n+2},x,-xq;q;zq \\ xq\sqrt{b},-xq\sqrt{b},x^2q^2\end{matrix}\right]$

By taking Take $\alpha_r = \frac{(x,\omega xq,\omega^2xq,x^3q^{n+4},q^{-n};q)_r}{(q,(xq)^{3/2},-(xq)^{3/2},x^{3/2}q^2,-x^{3/2}q^2;q)_r} q^r$ & $\delta_r = z^r$ in (4.5) and summing,

we get

(4.11) ${}_5\Phi_4\left[\begin{matrix}x,\ \omega xq,\ \omega^2xq,\ x^3q^{n+4},\ q^{-n}\ ;q,q \\ (xq)^{\frac{3}{2}},-(xq)^{\frac{3}{2}},x^{\frac{3}{2}}q^2,-x^{\frac{3}{2}}q^2\end{matrix}\right]_n$

$= z^{n+1}\frac{x^n[x^2q^4;q]_n[q;q]_n[x^3q^6;q^3]_n[xq^3;q]_{3m}}{[x^3q^4;q]_n[xq;q]_n[q^3;q^3]_m[x^2q^4;q]_{3m}}$

$+(1-z)\ {}_5\Phi_4\left[\begin{matrix}x,\ \omega xq,\ \omega^2xq,\ x^3q^{n+4},\ q^{-n}\ ;q,q \\ (xq)^{\frac{3}{2}},-(xq)^{\frac{3}{2}},x^{\frac{3}{2}}q^2,-x^{\frac{3}{2}}q^2\end{matrix}\right]$

By taking $\alpha_r = \frac{\left(a^{\frac{1}{3}},\omega a^{\frac{1}{3}},\omega^2a^{\frac{1}{3}},q\sqrt{a},aq^{n+1},q^{-n};q\right)_r}{(q,\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},q^2\sqrt{a};q)_r} q^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

(4.12) ${}_6\Phi_5\left[\begin{matrix}a^{\frac{1}{3}},\omega a^{\frac{1}{3}},\omega^2a^{\frac{1}{3}},q\sqrt{a},aq^{n+1},q^{-n};q,q \\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},q^2\sqrt{a}\end{matrix}\right]_n$

$$= z^{n+1} \frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(\sqrt{a})^{n-m}}{[aq;q]_n[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m}$$

$$+(1-z)_6\Phi_5\left[\begin{matrix} a^{\frac{1}{3}}, \omega a^{\frac{1}{3}}, \omega^2 a^{\frac{1}{3}}, q\sqrt{a}\ , aq^{n+1}, q^{-n}; q, qz \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{matrix}\right]$$

By taking $\alpha_r = \frac{(a, q\sqrt{a}, -q\sqrt{a}\,, b, c, d, a^2q^{1+n}/bcd, q^{-n}; q)_r}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n}; q)_r} q^r$ & $\delta_r = z^r$ in (4.5) and summing, we get

$$(4.13)\ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}\,, b, c, d, a^2q^{1+n}/bcd\,, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b\,, aq/c\,, aq/d\,, bcdq^{-n}/a\,, aq^{1+n} \end{matrix}; q; q\right]_n$$

$$= z^{n+1}\ \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}$$

$$+(1-z)\ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}\,, b, c, d, a^2q^{1+n}/bcd\,, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b\,, aq/c\,, aq/d\,, bcdq^{-n}/a\,, aq^{1+n} \end{matrix}; q; zq\right]$$

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