F(R) GRAVITY WITH K-ESSENCE SCALING RELATION AND COSMIC ACCELERATION

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Abstract

A modified gravity theory with $f(R) = R^2$ coupled to a dark energy lagrangian $L = -V(\varphi)F(X)$, $X = \nabla_\mu \varphi \nabla^\mu \varphi$, gives plausible cosmological scenarios when the modified Friedman equations are solved subject to the scaling relation $X(\frac{dF}{dX})^2 = Ca(t)^{-6}$. This relation is already known to be valid, for constant potential $V(\varphi)$, when L is coupled to Einstein gravity. φ is the k-essence scalar field and $a(t)$ is the scale factor. The various scenarios are: (1) Radiation dominated Ricci flat universe with deceleration parameter $Q = 1$. The solution for φ is an inflaton field for small times. (2) Q is always negative and we have accelerated expansion of the universe right from the beginning of time and φ is an inflaton for small times. (3)The deceleration parameter $Q = -5$, i.e., we have an accelerated expansion of the universe. φ is an inflaton for small times. (4)A generalisation to $f(R) = R^n$ shows that whenever $n > 1.780$ or $n < -0.280$, Q will be negative and we will have accelerated expansion of the universe. At small times φ is again an inflaton.The results remind us of other physical phenomena where existence of scaling relations signal some sort of universality for theories with different microscopic lagrangians. Here this is seen in the case of Einstein gravity and modified gravity theories.

Keywords: *Lagrangian, Hubble parameter, pressure, dark energy, ac- celeration.*

1. Introduction

Very often the existence of similar scaling relations in apparently different physi- cal systems signify the presence of some sort of universality. A physical system is usually typified by some lagrangian from which an action is constructed. There- fore, different actions describe different theories. However, if the same scaling relation is valid in theories with different actions, then it is possible that these theories may describe physical scenarios which have certain aspects in common. The motive of this paper is to show that in cosmology such a situation can occur in the context of dark energy where the dark energy is realised by $k-$ essence scalar fields φ.

A $k-$ essence theory coupled with a non-canonical lagrangian coupled to Ein- stein gravity is known to satisfy a scaling relation. Consider the usual Einstein- Hilbert action $S_{\mathrm{EH}} = \frac{1}{2k} \int d^4 x \sqrt{-g} R$ in presence of $k-$ essence fields φ with a non-canonical lagrangian $L = V(\varphi)F(X)$, $X = \nabla_\mu \varphi \nabla^\mu \varphi$. $\kappa \equiv 8\pi G$, G is the gravitational constant, g is the determinant of the metric and we work in units where $c = \hbar = 1$. Relevant literature on dark energy and $k-$ essence can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

For constant potential $V(\varphi)$, it follows from the resulting Friedman equations that $X(\frac{dF}{dX})^2 = Ca(t)^{-6}$ [15, 16, 17, 18]. In this work we investigate what happens if this same scaling relation is required to be valid when the same $k-$ essence lagrangian is coupled to modified gravity or $f(R)$ gravity theories where f is a general function of the Ricci scalar R [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

There are various ways of obtaining such theories. The modified Friedman equations for such $f(R)$ gravity theories were first obtained in [19]. We shall follow the notations of [23, 29]. In one approach, $f(R)$ theories of gravity are obtained by generalising the Einstein-Hilbert action into

$$S_{\mathrm{f}} = \frac{1}{2k} \int d^4 x \sqrt{-g} f(R) \tag{1}$$

where $f(R) = R$ gives the usual Einstein gravity.

We work in a Freidman-Lemaitre-Robertson-Walker metric with curvature constant $k = 0$:

$$ds^2 = c^2 dt^2 - a^2(t)[dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (2)$$

Adding a matter term S_M, the total action

$$S_f = \frac{1}{2k}$$

$$\int d^4 x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \varphi) \qquad (3)$$

where φ are generic matter fields. We shall consider a single scalar field only.   The field equations are [23, 24, 25, 26, 27, 28, 29]:

$$f'(R)\, R_{\mu v} - \frac{1}{2} f(R)\, g_{\mu v} - \nabla \mu \nabla v - \; g_{\mu v} \nabla^{\beta} \nabla_{\beta}]f'(R) = kT_{\mu v}$$
(4)

with

$$T_{\mu v} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu v}}$$
(5)

A prime denotes differentiation with respect to the argument. ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric. The 00 component of the field equations Eq.(4) gives the modified Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3f'(R)}\left(\frac{1}{2}[f(R) - Rf'(R) - Rf'(R)] - 3\left(\frac{\dot{a}}{a}\right)R\dot{f}''(R) = \frac{1}{3}k\rho$$
(6)

while the *ii* components give

$$2\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{f'(R)}\left(2\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) + \left(\dot{R}\right)^2 f'''(R) -$$

$$\frac{1}{2}[f(R) - Rf'(R)] = -kp$$
(7)

We first recall the relevant equations for the usual case

$f(R) = R\,[15, 16, 17, 18]$.

The energy density obtained from Eq.(6) is

$$\rho = (3H^2)/k$$
(8)

while Eq.(7) gives the pressure as

$$p = -\frac{2\ddot{a}}{ka} - \frac{H^2}{k} \tag{9}$$

Differentiating Eq.(8) with respect to time, and using Eq.(9) gives

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{10}$$

Now, as already stated before, the lagrangian of $k-$essence scalar fields φ is non-canonical and is of the form $L = p = -V(\varphi)F(X)$ where $V(\varphi)$ is the potential. The energy density is $\rho = V(\varphi)[F(X) - 2XF_X]$ with $F_X \equiv \frac{dF}{dX}$. Substituting these values of ρ and p in Eq.(10) and taking $V(\varphi) = $ constant one gets

$$\left(\frac{dF}{dX} + 2X\frac{d^2F}{dX^2}\right)a\frac{dX}{da} + 6\frac{dF}{dX}X = 0 \tag{11}$$

This equation can be integrated to give the scaling relation $[15, 16, 17, 18]$

$$X\left(\frac{dF}{dX}\right)^2 = Ca(t)^{-6} \tag{12}$$

C is a constant.

Using Eq.(12) and the zero-zero component of Einstein's field equations an expression for the lagrangian for the $\overline{k}$ essence field can be obtained for a FLRW metric. This has been elaborately described in [17] where the scaling relation Eq.(12) has been used to eliminate F_X. For a homogeneous scalar field $\varphi(t)$ the lagrangian is [17]

$$L = -c_1\dot{q}^2 - c_2 V(\varphi)\dot{\varphi}\, e^{-3q} \qquad (13)$$

where $q(t) = \ln a(t)$, $c_1 = 3(8\pi G)^{-1}$, $c_2 = 2\sqrt{C}$. The new lagrangian has two generalised coordinates $q(t)$ and $\varphi(t)$. q has a standard kinetic term while φ does not have a kinetic part. There is a complicated polynomial interaction between q and φ and φ occurs purely through this interaction term.

We now consider the case for $f(R) = R$. Then Eq.(6) gives for the energy density

$$\rho = \frac{3}{k}H^2 + \frac{R}{4k} + \frac{3H}{k}\frac{\dot{R}}{R} \qquad (14)$$

while Eq.(7) gives the pressure (or lagrangian)

$$p = -\frac{2\ddot{a}}{ka} - \frac{H^2}{k} - \frac{2H\dot{R}}{kR} - \frac{\ddot{R}}{kR} - \frac{R}{4k} \qquad (15)$$

Using $\frac{\ddot{a}}{a} = \dot{H} + H^2$ we have

$$\rho + 3\dot{H}(\rho + p) = \frac{\dot{R}}{4k} + 3\dot{H}\frac{\dot{R}}{kR} - 3H\frac{\dot{R}^2}{kR^2} + 3H^2\frac{\dot{R}}{kR} \qquad (16)$$

It is readily seen that Eq.(16) reduces to Eq.(10) if the right hand side vanishes i.e.

$$\dot{R}\left[R^2 + 12H\dot{R} - 12H\dot{R} + 12H^2R\right] = 0 \tag{17}$$

i.e. if either $\dot{R} = 0$ or $\dot{R}\left[R^2 + 12H\dot{R} - 12H\dot{R} + 12H^2R\right] = 0$. Now $R = 6[\dot{H} + 2H^2]$ so that $\dot{R} = 6[\ddot{H} + 4H\dot{H}]$. Using this in Eq.(17) we have two scenarios for which Eq.(16) reduces to Eq.(10), *viz.*:

$$\ddot{H} + 4H\dot{H} = 0 \tag{18}$$

and

$$3\dot{H}^2 - 2H\ddot{H} + 2H^2\dot{H} + 8H^4 = 0 \tag{19}$$

1. Solutions for Hubble parameter

Now we know the conditions (i.e. Eq. (18) or Eq. (19) that the Hubble parameter has to satisfy so that the scaling relation Eq. (12) is valid in a modified gravity theory with $f(R) = R^2$.

1.1 Solutions for $\dot{R}=0$

A first integral of Eq.(18) gives $R = 6(\dot{H} + 2H^2) = 6A$ where A is a constant.

Case 1: $A = 0$, i.e.$R = 0$ (Ricci flat):

If the constant A is chosen to be zero then we have $R = 6(\dot{H} + 2H^2) = 0$ so

that the emergent spacetime is Ricci flat. A simple integration gives

$$H = 1/2t .$$

This implies that the scale factor is $a(t) \sim t^{1/2}$ and the cosmology is that of the

well known radiation dominated universe. From Eq.(14), the energy density is

$$\rho = 3/4kt^2 \tag{20}$$

because in this case $R = 6[\dot{H} + 2H^2] = 0$. The lagrangian or pressure can be

obtained from Eq.(15) as

$$p = 1/4kt^2 \tag{21}$$

To determine the k-essence scalar field φ recall [17]. Take $V(\varphi) = V = constant$

then one has

$$L = -V(\varphi)F(X) = -c_1\left(\frac{\dot{a}}{a}\right)^2 - c_2 V\dot{\varphi}a^{-3}$$

$$= c_1\left(\frac{1}{4t^2}\right) - c_2 V\dot{\varphi}a^{-\frac{3}{2}} \tag{22}$$

where   and $c_2 = 2\sqrt{C}$. It is to be noted that the equation Eq.(22) incorporates

$c_1 = \frac{3}{k}$

the scaling relation. Eq.(21) and Eq.(22) then give the solution for the k−essence scalar field as

$$a = \frac{1}{32\pi G \sqrt{CV}} t^2 \tag{23}$$

where we have put all integration constants to zero. Now $t = \frac{t}{t_0}$ where t_0 is the present epoch and so t is always less than unity. Now for $0 < t < 2$,

$\ln t \sim (t-1)$. So $t^{1/2} = e^{(1/2)\ln t} \sim e^{(1/2)(t-1)} \sim 1 + t$. Hence for small times

$\varphi(t) \sim 1/(64\pi G \sqrt{CV}) + t/(64\pi G \sqrt{CV})$ and this is like the scalar field in "chaotic inflation"as in [33].

The deceleration parameter $Q = -(a\ddot{a})/\dot{a}^2 = 1$ while the equation of state parameter $\omega = p/\rho = 1/3$. So we have a Ricci at $(R = 0)$ decelerating universe with radiation domination.

The interesting aspect is that this has been obtained from a modified gravity theory having a *dark energy constituent satisfying a scaling relation that is also satisfied in Einstein gravity.*

Case 2: $A \neq 0$ i.e. $R \neq 0$:

If $A \neq 0$, then a solution for the Hubble parameter is (choosing an integration constant to be zero)

$$\sqrt{\tfrac{A}{2}}\tanh[\sqrt{2At}] \tag{24}$$

then   The scale factor is The scale factor is then
$$H(t) =$$

$$a(t) = cosh^{\frac{1}{2}}[\sqrt{2At}] \tag{25}$$

and the deceleration parameter is

$$Q = 1 - 2coth^2[\sqrt{2At}] \tag{26}$$ Hence

the deceleration parameter is always negative as $1 \leq coth t \leq \infty$ for $t \geq 0$. So we have accelerated expansion of the universe right from the beginning of time.

Pressure p now is

$$p = -(1/k)[2\dot{H} + 3H^2 + R/4] \tag{27}$$

Putting in $R = 6A$ and $2\dot{H} = 2A - 4H^2$ gives

$$p = H^2/k - 7A/2k = (A/2k)\,tanh^2[\sqrt{2At}] - 7A/2k \tag{28}$$

where we have put in H from Eq.(24).

$$p = -c_1(A/2)\,tanh^2[\sqrt{2At}] - c_2 V\dot{\varphi}cosh^{-3/2}[\sqrt{2At}] \tag{29}$$

Alternatively, using Eq.(22) we have

$$(d\varphi/dt) = (7A/2kc_2 V)cosh^{3/2}\left[\sqrt{2At}\right] - (A/2\,c_2 V)(1/k + c_1)$$

$$tanh^2\left[\sqrt{2At}\right]cosh^{3/2}\left[\sqrt{2At}\right] \quad \text{which upon integration gives}$$

$$\varphi(t) = \left(\sqrt{A}/\sqrt{2}\,kc_2 V\right)sin\,h\left[\sqrt{2At}\right]cosh^{1/2}\left[\sqrt{2At}\right] -$$

$$\left(15\,\sqrt{A}/\sqrt{2}\,kc_2 V\right)iF((i\sqrt{2At}/2)|2)/3 + constant \qquad (30)$$

where the elliptic function F is defined as $F(\alpha|m) = \int_0^\alpha (1 - msin^2\theta)^{-1/2}\,d\theta$.

Note that for $\alpha \to 0, F \to 0$. Therefore, in the early universe where $t \equiv t/t_0 \ll 1$, t_0 being the present epoch, the F term in Eq.(30) can be safely ignored. Then for early for early times $\varphi(t)$ again qualifies for an inflationary field [33] since the Eq.(30) gives $\varphi(t)\sim const. + (A/kc_2 V)t$

1.2 Solutions for $\dot{R} \neq 0$

If $\dot{R} = 0$, then Eq.(19) should hold. The equation Eq.(19) is readily solved with the ansatz $H = \alpha/t$ with α a constant. Using this ansatz ,Eq.(19) gives for $\alpha = 0$

$$8\alpha 2 - 2\alpha - 1 = 0 \qquad\qquad (31)$$

This is a quadratic in α and for real α the solutions are $\alpha = (1/2), (-1/4)$ so that the Hubble parameter solutions corresponding to the two values of α are

$$H = \quad 1/2t;$$

$$-1/4t \qquad\qquad (32)$$

Of these two solutions , we have already encountered the first one. So we consider the other solution only, i.e. $H = -1/4t$. This gives the solution for the scale parameter as

$$\alpha(t) = const. \, t^{-1/4} \qquad\qquad (33)$$

The deceleration parameter is now $Q = 5$. So again we have an accelerated expansion owing its origin to dark energy.

The Ricci scalar is $R = 6[\dot{H} + 2H^2] = 9/4t^2$.

The energy density using (14)   now is

$$\rho = 9/4kt^2 \qquad (34)$$

while from Eq.(15) the pressure is obtained as

$$p = -33/4kt^2 \qquad (35)$$

As before, equating Eq.(35) to the lagrangian gives

$$-33/4kt^2 = -c_1(\dot{a}/a)^2 - c_2 V\dot{\varphi}a^{-3}$$

$$= -c_1(1/16t^2) - c_2 V\varphi(\dot{const.})t^{-3/4} \qquad (36)$$

Proceeding as before, a solution for the dark energy scalar field is obtained as

$$\dot{\varphi}(t) = (const.)t^{-1/4} \qquad (37)$$

Note that we can write $\varphi(t) \equiv (const.)e^{-(1/4)lnt} \sim e^{-(\frac{1}{4})(t-1)}$ for $0 < t < 2$.

In our case this is always true as $t \equiv t/t_0$ always lies between 0 and 1. Therefore $\varphi(t) \sim e^{-1/4(t-1)} = 1 - t/4$ again qualifes for an inationary field [33].

2. GENERALISATION TO $f(R) = R^n$

We shall now consider the general case for $f(R) = R^n$ and for convenience take $\kappa = 1$. Then $f'(R) = nR^{(n-1)}$; $f''(R) = n(n-1)R^{(n-2)}$; $f'''(R) = n(n-1)(n-2)R^{(n-3)}$ and Eq.(4) becomes

$$\rho = 3H^2 + (n-1)R/2n + 3(n-1)\dot{R}H/R \qquad (38)$$

so that

$$\dot{\rho} = 6H\dot{H} + (n-1)\dot{R}/2n + 3(n-1)\dot{R}\dot{H}/R + $$
$$3(n-1)\ddot{R}H/R - 3(n-1)\dot{R}^2H/R^2 \qquad (39)$$

The pressure density is given by

$$p = -2\dot{H} - 3H^2 - (n-1)(2H\dot{R} + \ddot{R})/R - $$
$$(n-1)(n-2)\dot{R}^2/R^2 - (n-1)R/2n \qquad (40)$$

From Eq.(38) and Eq.(40) we have

$$3H(\rho + p) = -6H\dot{H} + 3(n-1)\dot{R}H^2/R - 3(n-1)\ddot{R}H/R$$
$$-3(n-1)(n-2)\dot{R}^2H/R^2 \qquad (41)$$

Using Eq.(39) and Eq.(41) gives the equation of continuity

$$\dot{\rho} + 3H(\rho + p) = (n-1)\dot{R}/2n + (3(n-1)\dot{R}/R)[\dot{H} + H^2 - (n-1)\dot{R}H/R] \quad (42)$$

For the scaling relation Eq.(12) to be valid, the right hand side of Eq.(42) must vanish. This gives (note that $n \neq 1$)

$$\dot{R}[(1/2n) + (3/R)(\dot{H} + H^2 - (n-1)\dot{R}H/R)] = 0 \qquad (43)$$

There are two possibilities:

First $\dot{R} = 6[\ddot{H} + 4H\dot{H}] = 0$ and $[(1/2n) + (3/R)(\dot{H} + H^2 - (n-1)\dot{R}H/R)] \neq 0$. This gives a solution $H(t) = 1/2t$. This we have already encountered and we have a radiation dominated universe.

The other solution is obtained when $\dot{R} \neq 0$. Then

$$[(1/2n) + (3/R)(\dot{H} + H^2 - (n-1)\dot{R}H/R = 0 \qquad (44)$$

Substituting $R = 6[\dot{H} + 2H^2]$ and $\dot{R} = 6[\ddot{H} + 4H\dot{H}]$ in Eq.(44) and solving for

H we get $H = (-2n^2 + 3n + 1)/(2 + n)t = (u/t)$ $\qquad$ (45)

where $u = (-2n^2 + 3n + 1)/(2 + n)$.

So the scale factor is obtained as

$$a(t) = t^u \qquad (46)$$

The deceleration parameter

$Q = -[1 + H/H^2]$ now is

$$Q = -[1 - (2 + n)/(-2n^2 + 3n + 1)] \qquad (47)$$

For late time acceleration of the universe Q should always be negative i.e., the term within third brackets in Eq.(47) must be positive i.e.,

$1 - (2 + n)/(-2n^2 + 3n + 1) > 0$ or $(2 + n)/(-2n^2 + 3n + 1) < 1$.

This means that whenever $n < (1/4)(3 - \sqrt{17}) = -0.280$ $\qquad$ or

$n > (1/4)(3 + \sqrt{17}) = 1.780$ there will be accelerated expansion of the universe.

We now determine the k essence scalar field φ. The lagrangian or pressure

is

$$p = -2\dot{H} - 3H^2 - (n-1)(2H\dot{R} + \ddot{R})/R - (n-1)(n-2)\dot{R}^2/R^2 - (n-1)R/2n$$

$$(48)$$

Now Therefore $\quad R = 6\big[\dot{H} + 6u(2u-1)/t^2$

$\dot{H} = -(u/t^{2)}. \; 2H^2\big] =$

$$\dot{R} = -(12u(2u-1)/t^3$$

$$\ddot{R} = (36u(2u-1)/t^4$$

Using these expressions in Eq.(48) we get

$$p = (-36n^4 + 84n^3 - 61n^2 + 14n - 1)/(n^2 + 4n + 4)t^2 = v/t^2 \quad (49)$$

4   3   2

where $= (-36n^4 + 84n^3 - 61n^2 + 14n - 1)/(n^2 + 4n + 4)$. Alternatively, the $k-$ essence lagrangian (pressure) Eq.(22) gives

$$L = -c_1(u^2/t^2) - c_2V(d\varphi/dt)\,t^{-3u} \qquad\qquad (50)$$

Equating Eq.(49) and Eq.(50) and solving for $\varphi(t)$ gives

$$\varphi(t) = A - (v + c_1 u^2)t^{(3u-1)}/c_2 V(3u-1)$$

$$= A - Bt^{(-6n^2+8n-1)/(2+n)} \qquad\qquad (51)$$

where A is a constant of integration and $B = (v + c_1 u^2)/(c_2 V(3u-1)$. Note that we can write $\beta = (-6n^2 + 8n - 1)/(2 + n)$

$$\varphi(t) = A - Bt^\beta = A - Be^{lnt^\beta}$$

$$= A - Be^{\beta lnt}$$

$$\sim A - Be^{\beta(t-1)}$$

$$\sim (A - B - B\beta) - B\beta t \qquad\qquad (52)$$

$$or\ \varphi(t) = constant - B\beta t \qquad\qquad (53)$$

So φ at small times is again like an inflationary field [9].

3. CONCLUSION

The importance of this work is that Einstein gravity and certain $f(R)$ grav- ity theories, coupled to the same non-canonical k-essence lagrangian, can lead to similar cosmological scenarios when a certain scaling relation involving the k essence fields is satisfied in both regimes. So although the underlying gravity theories are different, the cosmological scenarios are realistically similar. This observation is in tune with other branches in physics where existence of scaling relations signify various genre of universality for theories with different micro-scopic lagrangians. Moreover, as the said scaling relation involves the derivatives of

the the dark energy scalar fields only, it may be conjectured that a major con-tribution to the cosmological consequences come from the k essence fields. In this work, we first discuss a modified gravity theory with $f(R) = R^2$ coupled to a non-canonical dark energy lagrangian $L = V(\varphi)F(X)$, with $X = \nabla_\mu \varphi \nabla^\mu \varphi$ and $V(\varphi)$ a constant. The modified Friedman equations are subjected to the constraint $X(dF/dX)^2 = Ca(t)^{-6}$ and solutions obtained for the scale factor $a(t)$. The deceleration parameter Q and the k - essence scalar field φ are then deter- mined. The following cosmological scenarios are obtained:

(a)Radiation dominated Ricci flat universe with deceleration parameter $Q = 1$. The scalar field takes the form of an inflaton field for for small times.
(b)The deceleration parameter is always negative and we have accelerated ex-pansion of the universe right from the beginning of time. Here also the scalar field is similar to an inflaton field for small times.
(c)The deceleration parameter $Q = 5$ and again we have an accelerated ex-pansion of the universe with the scalar field akin to an inflaton field for small times.
(d) A generalisation to $f(R) = R^n$ is then discussed. It is shown that whenever $n > 1.780$ or $n < 0.280$, Q will be negative and we will have accelerated expansion of the universe. At small times the scalar field again behaves like an inflaton.

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