# RECREATIONAL MATHEMATICS

## Part One

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**Abstract:**

Over centuries mathematicians have generated a wealth of rigorous and high level mathematics that is the armoury of pure mathematicians. But there is an interesting segment of mathematics that can justifiably be consigned to a different realm, which is the world of recreational mathematics. In this paper we will visit a few interesting areas of this fascinating domain.

**Keywords**: *Rigorous, armoury, realm, recreational*

We begin with an elementary quiz.

1. A man leaves home one morning for a whiff of fresh air. He has just readied his breakfast, and laid it on the table .His intention is clear, not to eat his breakfast till he has completed a 3 km walk. He leaves the door wide open, behind him and starts walking in the southward direction. After walking a kilometre, he takes a left turn, and heads eastward. He finishes his eastward sojourn and again takes a left turn,now facing North. After walking a kilometre north ward, he finds himself back home, and a huge bear gulping down his breakfast.

The question is: What is the colour of the bear??

We will wait patiently, and meet the answer at the end of this note.

2. The second story is more paradoxical. It is about an ancient traveller who gets tired of walking under the hot sun and sits down under the shade of a huge banyan tree. Under the shade of this huge tree he finds a barber shaving a man. When the barber is done with his job, and as he shows signs of leaving, the traveller asks him a simple question: “What do you do for a living”?

The barber politely replies: “ Well,I’m the village barber. You see, I shave all those who don’t shave themselves”. Undoubtedly an innocent answer. Yes indeed, till we start examining the barber’s plight. Does the barber shave himself?? Or, doesn’t he?

To start with, let us assume that the barber DOES shave himself.

He then belongs to the set of people who shave themselves, and are not shaved by the barber, So, we conclude that the barber does not shave himself !

We now assume the opposite. In other words we assume that the barber DOES NOT shave himself. He immediately gets into the category of people who do not shave themselves, and are shaved by the barber. So the barber DOES shave himself. Now this is paradoxical: when we assume he shaves himself we conclude he doesn't. And when we assume he doesn't we conclude he does.

Will growing a beard help him?? Well, not really.

In modern set theory the above paradox has found a prominent place and goes by the name "Russel's paradox",named after the famous Cambridge mathematician Bertrand Arthur William Russell.( 1872- 1970)

[ After he completed his studies in Cambridge Bertrand Russell began questioning the foundations of Mathematics, especially that of Euclidean geometry. He could not tolerate the idea of defining a line by using the concept of a point OR the idea of defining a point by the intersection of two or more lines . Such simple questions goaded him to domains of philosophy, to mathematical philosophy in particular.

His book titled "An introduction to Mathematical Philosophy" churns around some simple ideas from a philosophical point of view, and is an interesting read.]

In a mathematical way this paradox is summarised in set theoretic language as:

If R is a member of itself it is not

& when R is not a member of itself it is !

3. We enter ,for a moment, into the field of number triplets {x,y,z} which are such that $x^2 + y^2 = z^2$

These are known as Pythagorean triples.

If we confine ourselves to positive integers, then the smallest set of three positive integers forming a Pythagorean triple is {3,4,5}

There are others like {5,12,13},{16,63,65},{8,15,17}…the list goes on.

It is quite interesting to note that the the first algebraic identity that we learn in middle school comes in pretty handy in generating a plethora of these Pythagorean triples. We are referring to the identity: $( A + B )^2 = ( A - B )^2 + 4AB$

All we do is to replace A by $m^2$ $and\ B\ by\ n^2$

And we look at the same relation, which looks slightly altered:

$$( m^2 + n^2)^2 = ( m^2 - n^2 )^2 + ( 2mn )^2$$

We then simply assign integral values to m and n (without any restrictions imposed on their relative magnitude).

If we choose,quite arbitrarily, m=4 and n = -3, we get,

$25^2 = 7^2 + 24^2$, which is a very familiar result. The important point to be appreciated is that an infinitude of these triples can be created by varying the values of m and n, and the list would be endless.,….

4. We now take look at a problem that is easy to state as well as to solve, but with interesting features.

We consider a line segment AC,whose length can be taken as unity, without loss of any generality). Embedded between A and C is a point, which we christen as B. We carefully position B so that it is to the right of the midpoint of AC.( In other words the length of segment AB is larger than the length of segment BC).

If AB is x units, then BC is evidently ( 1 –x ) units.

We try to position B in such a way that $\frac{AC}{AB} = \frac{AB}{BC}$

If we insert the known values of the segments in the above relation we get: $\frac{1}{x} = \frac{x}{1-x}$, which is equivalent to the quadratic

$$x^2 + x - 1 = 0,$$

The two roots that emerge are : $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$.

Since we are considering positive lengths we reject the second root, which is negative. The positive value of x is 0.618033988.

(The point B that cuts up the unit line in the fashion described above is called the golden section,and the ratio called "golden ratio")

Here we may ask a simple question: SO WHAT?

Let's wait a while to build up a link between the above ratio and the following sequence: 1,1,2,3,5,8,13,21,34,55,89,144,233,377….

(This is the familiar Fibonnacci sequence of numbers in which every number, after the second, is the sum of the TWO numbers immediately preceding it).Using symbols for recurrence formulae we write: $u_{n+2} = u_{n+1} + u_n$

Dividing through by $u_{n+1}$, we get: $\frac{u_{n+2}}{u_{n+1}} = 1 + \frac{u_n}{u_{n+1}}$

If we study the terms we find that the left side is the ratio of the "next" term to the "current " term, considering $u_{n+1}$ as the current term. On the right side the terms involving n give the ratio of a "previous" term to the current term .

If we assume that for very large values of n,the left hand ratio has a limiting value μ ,then we can re-cast the equation in the form: $\mu = 1 + \frac{1}{\mu}$, which is equivalent to

$\mu^2 = \mu + 1$, yielding a value of $\mu = \frac{\sqrt{5}-1}{2}$, the value of which is 0.618033988,a decimal fraction with no integral part. It is curious that its reciprocal has THE SAME fractional part as the number itself!

We also recognise this number to be the golden ratio.

Another remarkable property of the Fibonacci series is that if a slice of any three consecutive Fibonacci numbers ( $u_n, u_{n+1}, u_{n+2}$) is cut out, the relation ${u_{n+1}}^2 = u_n . u_{n+2} \pm 1$ …………………………….(1)

holds good ,as can be quickly verified.

Stated simply : The square of the middle term exceeds,or falls short of, the product of the two numbers flanking the middle number.For example $5^2 = (3).(8) + 1$ and $8^2 = (5).(13) - 1$

[Fibonacci's actual name was Leonardo of Pisa. His father was called Bonaccio.Fibonacci( fil+ Bonaccio) literally translates to " son of Bonaccio"]

According to Leonardo,if a male-female pair of adult rabbits is placed in an enclosure and allowed to breed then at the end of twelve months the enclosure will have 377 pairs of rabbits{ it being assumed that each pair of offsprings is a male-female pair,that starts to breed a pair( every month) after attaining the age of two months}.

A small table can be constructed to reveal that the number of pairs at the ends of month numbers 1,2,3,4,5..are respectively 1,1,2,3,5,…

which correspond to the first five Fibonacci numbers.

Using the theory of recurrence relations it has been demonstrated that the $n^{th}$ Fibonacci number $F_n$ , is captured in the relation

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

If we remember the greatest integer function of x,{denoted by [x],as being the greatest integer not exceeding x},then for ANY Fibonacci number $F_n$ , the next Fibonacci number is given by( the integral part of) the number [$\frac{F_n + 1 + \sqrt{5F_n^2}}{2}$] , rounded off to the NEAREST Integer less than the computed number……………………………………(2)

As an illustration, the 4$^{th}$ Fibonacci number is 3.Substituting this value into the above expression we get 5.3541. Rounding off to the nearest lower integer we get the 5$^{th}$ Fibonacci number to be 5.

Another interesting feature of this expression is the fact that one can start with ANY number as the FIRST number. Let us take the number 9,and compute the "next" number, which turns out to be 15,and the subsequent numbers are 24,39,63,102.It is interesting to note that the recursion relation of the original Fibonacci series; BUT the property (1) does not hold,but is modified. We observe that in any trio, the square of the middle member exceeds by 9,the product of the flanking numbers. For example

$$15^2 = (9).(24) + 9\ ,\ \ 24^2 = (15).(39) - 9\ ,\ \ 39^2 = (24).(63) + 9$$

And so on. (A DIFFERENCE of 9 is obtained, which equals the FIRST number of the series!! Is this a coincidence? We will soon find out.)

By choosing the first number as 7, the sequence that emerges by using the relation (2), we get the following sequence:7,11,18,29,47,76,123,199….

Using the numbers of the sequence, and reasoning as before we find that ${T_{n+1}}^2 = (T_n).(T_{n+2}) \pm 5$.

So, we can conclude that the difference is CONSTANT but not necessarily equal to the FIRST number chosen.

5. We will end our deliberations by examining how our own statements turn their heads on us. Socrates once said" All Greeks are liars".This directly proves that all Greeks are NOT liars!

Socrates is a Greek himself,so he is a liar. If his statement above is a lie,then all Greeks are NOT liars !!

We have one last job to finish. In the very beginning we promised to make up our minds on the colour of the bear. The mans house is at the North pole!! If one starts walking southward he is on a line of longitude. When he takes a left turn to the East he picks up a line of latitude, and upon turning left he is on another line of longitude that carries him back to his own house where a polar bear is hurriedly gulping down his breakfast.

## CONCLUSION

The above is a very brief introduction to this wondrous world of recreational mathematics. In subsequent parts the author intends covering other areas .

## REFERENCES

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