RINGS WITH THE DUAL OF THE ISOMORPHISM THEOREM

[1]Iqra Liaqat, [2]Kaushef Salamat

[1]Department of Mathematics, Punjab University New Campus,

Lahore District-75500, Pakistan.

[2]Department of Mathematics, Lahore Garrison University,

Lahore District-75500, Pakistan

Email: pro.iqra.14@gmail.com

Abstract:

A ring R satisfies the dual of the isomorphism theorem if $R/Ra \cong 1(a)$ for all elements a of R, where 1(a) denotes the left annihilator. We call these rings left morphic. Examples include all unit regular rings and certain left uniserial local rings. We show that every left morphic ring is right principally injective, and use this to characterize the left perfect, right and left morphic rings.

Keywords: *Primary 16E50, secondary 16U99, 16S70.*

Introduction:

It is a well-known theorem of Erlich [5] that a map α in the endomorphism ring of a module M is unit regular if and only if it is regular and $M/\operatorname{im}(\alpha) \cong \ker(\alpha)$. Our focus is on the case $M = {}_RR$, so if $\alpha = .a: {}_RR \to {}_RR$ is right multiplication by the element $a \in R$, the condition becomes $R/Ra \cong 1(a)$ where $1(a)$ denotes the left annihilaror. We say that the ring R is left morphic if every element satisfies this condition. We begin (Theorem 9) by characterizing the left morphic , local rings with nilpotent radical (and call these rings left 'special'); in particular we show that these rings are all left artinian. We show (Theorem 29) that a semiperfect left morphic ring is a finite product of matrix rings over local left morphic rings, we use this result to characterize (in Theorem 35) the left perfect , left and right morphic rings as the finite products of matrix rings over left and right 'special' rings.

Along the way, we show (Theorem 24) that every left morphic ring is right principally injective [11]. With this we see that the left morphic rings R with ACC on right annihilators are left artinian , and have the property that eRe is left 'special' for every local idempotent e in R (Theorem 31). In fact, we show that if R is left morphic then eRe is also left morphic for every idempotent $\in R$. However, we give examples to show that the matrix ring $M_n(R)$ need not be left morphic, and so 'left morphic' is not a Morita invariant (unlike 'unit regular').

Throughout this paper every ring R is associative with unity and all modules are unitary. If M is an R-module we write $J(M)$, $\operatorname{soc}(M)$ and $Z(M)$ for the Jacobson radical, the socle ,and the singular submodule of M, respectively . We often abbreviate $J(R) = J$, and we write $U = U(R)$ for the group of units of R. A submodule $N \subseteq M$ is said to be an essential submodule (written $N \subseteq^{ess} M$) if $N \cap$

$K \neq 0$ for every nonzero submodule K of M. We denote left and right annihilators of a subset $X \subseteq R$ by $1(X)$ and $r(X)$ respectively , and we write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n.

1. Examples:

If R is a ring , an element a in R is called left morphic if $R/Ra \cong 1(a)$. The ring itself is called a left morphic ring if every element is left morphic. These rings are our primary interest in this paper, and the following lemma will be used frequently.

Lemma 1. The following are equivalent for an element a in a ring R:

1. a is left morphic , that is $R/Ra \cong 1(a)$.
2. There exists $b \in R$ such that $Ra = 1(b)$ and $1(a) = Rb$.
3. There exists $b \in R$ such that $Ra = 1(b)$ and $1(a) \cong Rb$.

Proof:

Given (1), let $\sigma: R/Ra \to 1(a)$ be an isomorphism , and put

$b = (1 + Ra)\sigma$.Then $Rb = im(\sigma) = 1(a)$ because σ is onto, and $1(b) = Ra$ because σ is one-to-one . Hence $(1) \Rightarrow (2), (2) \Rightarrow (3)$ is clear. But if (3) holds then $R/Ra = R/1(b) \cong Rb \cong 1(a)$.

An elementary argument using condition (2) in Lemma 1 shows that

Example 2.

A direct product of rings is left morphic if and only if each factor is left morphic.

It is clear that every unit and every idempotent in a ring R, is left (and right) morphic. The following lemma will be referred to several times.

Lemma 3.

If a is a left morphic element in a ring R, the same is true of au and ua for every unit u in R.

Proof:

Choose $b \in R$ such that $Ra = 1(b)$ and $Rb = 1(a)$. Then $R(ua) = Ra = 1(b) = 1(bu^{-1})$ and $R(bu^{-1}) = 1(a)u^{-1} = 1(ua)$, so ua is left morphic. Again, $R(au) = 1(b)u = 1(u^{-1}b)$ and $R(u^{-1}b) = Rb = 1(a) = 1(au)$.

An element a in a ring R is called (unit) regular if $aba = a$ for some (unit) $b \in R$, and the ring R is called a (unit) regular ring if every element has the property. If a is unit regular, say $aua = a$ where u is a unit, and if $e = ua$, then $a = u^{-1}e$ is left and right morphic by Lemma 3 because $e^2 = e$. This gives a very simple proof of the following important known result of Erlich [5].

Example 4.

Every unit regular ring is left and right morphic.

The converse of the assertion in Example 4 is false: The ring $\mathbb{Z}_4$ of integers modulo 4 is left and right morphic by Lemma 7 below. However, we can ask:

Question . If R is left and right morphic and $J = 0$, is R (unit) regular?

To show that the answer is 'yes' it is enough to show that R is regular by the following theorem of Erlich [5]. For completeness, we include a simple proof of this result that provides an explicit formula for the middle unit.

Proposition 5. (Erlich). If $a \in R$ is both regular and left morphic , then a is unit regular.

Proof:

Let $axa = a$, and assume that $Ra = 1(b)$ and $1(a) = Rb$ for some

$b \in R$. Write $u = xax + b$, so that $aua = a$. To see that u is a unit, observe first that $1 - ax \in 1(a) = Rb$, say $1 - ax = yb, y \in R$. If we write $v = a + y(1 - xa)$, it is a routine matter to verify that $vu = 1$. To show that $uv = 1$, it suffices to show that $1(u) = 0$. So suppose that $ru = 0$, that is $r\,xax + rb = 0$. Right multiplication by a gives

$rxaxa = 0$, whence $rxa = 0$. It follows that $rb = 0$, so $r \in 1(b) = Ra$,

Say $r = ta$. But then $0 = rxa = taxa = ta = r$, as required.

Camillo and Yu [3] show that every unit regular ring is clean (where a ring is called clean if every element is the sum of a unit and an idempotent). Hence we ask:

Question:

Is every left and right morphic ring clean?

Note that Camillo and Yu show that every semiperfect ring is clean, so Example 8 below is clean but not right morphic.

The following properties of left morphic rings will be used repeatedly. Recall that a ring R is called directly finite if $ab = 1$ in R implies that

$ba = 1$.

Proposition 6. If $a \in R$ is left morphic, the following are equivalent:

(1) $1(a) = 0$.

(2) $Ra = R$.

(3) $a \in U$.

In particular, every left morphic ring is directly finite.

Proof:

Choose $b \in R$ such that $Ra = 1(b)$ and $1(a) = Rb$.Then $1(a) = 0$ if and only if $b = 0$, and $Ra = R$ if and only if $b = 0$. This proves that

$(1) \Leftrightarrow (2)$, and these certainly are equivalent to (3). To see that R is directly finite, suppose that $uv = 1$ in R. Then $1(u) = 0$ so u is a unit

(by (1)$\Rightarrow$ (3)) and $v = u^{-1}$.

Thus a polynomial ring R[x] is never left (or right) morphic because $1(x) = 0$, and the only left morphic domains are the division rings.

The next lemma gives another source of examples of left morphic rings. Recall that a ring R is callrd local if it has a unique maximal left (or right) ideal, equivalently if R/J is a divison ring, equivalently if $R - J$ consists of units.

Lemma 7. If a ring R has a unique left ideal , $L \neq 0, R$ then R is left morphic.

Proof:

Let $a \in R$. To show R is left morphic, we must show that 'a' is left morphic. Since R has unique left maximal ideal i.e, R is local. So let

$a \neq 0 \in L$. Then $= Ra$. Also, as $a \neq 0 \Rightarrow I(a) \neq R$. (by definition)

We will show $I(a) \neq 0$. Suppose $I(a) = 0 \Rightarrow f: R \to Ra$ defined by

$f(r) = ra$ is an isomorphism.

$\Rightarrow R \cong Ra \Rightarrow 0 \subseteq La \subset L$

Which is contradiction.

Hence , $I(a) \neq 0 \Rightarrow I(a) = L$.

$\Rightarrow I(a) = Ra \Rightarrow$ a is left morphic.

$\Rightarrow$ R is left morphic.

Example 8.

Let F be a field with an isomorphism $x \to \bar{x}$ from F to a subfield $\bar{F} \neq F$. Let R denote the left F-space on basis $\{1,c\}$ where $c^2 = 0$ and $cx = \bar{x}c$ for all $x \in F$. Then R is a left artinian , local, left morphic ring that is not right morphic .

Proof:

One verifies that $Rc = Fc = J$, and that $0, J$ and R are the only left ideals of R. Thus R is local, and it is left morphic by Lemma 7. Choose

$y \in F - \bar{F}$ and put $a = yc$; we show that a is not right morphic. Suppose that $b \in R$ exists such that $aR = r(b)$ and $r(a) = bR$. Then $0 \neq b \in J$, say $b = xc$ where $0 \neq x \in F$. Hence $r(b) = Fc = J$, so

$aR = J$. In particular $c \in aR$, so $c = y\bar{z}c$ for some $z \in F$. It follows that

$y = \bar{z}^{-1} \in \bar{F}$, contrary to our choice .

The ring in Example 8 turns out to be a prototype for all local, left morphic rings with nilpotent Jacobson radical.

Theorem 9.

The following conditions are equivalent for a ring R:

(1) R is left morphic, local and J is nilpotent.

(2) R is local and $J = Rc$ for some $c \in R$ with $c^n = 0, n \geq 1$.

(3) There exists $c \in R$ and $n \geq 1$ such that $c^{n-1} \neq 0$ and

$R \supset Rc \supset Rc^2 \supset \cdots \supset Rc^n = 0$ are the only left ideals of R.

(4) R is left uniserial of finite composition length.

(5) There exists $c \in R$ such that $c^n = 0, n \geq 1$, and

$R = \{uc^k | k \geq 0, u \in U\}$.

If c is as in (3) then

(a) $1(c^k) = Rc^{n-k}$ and $Rc^k - Rc^{k+1} = Uc^k$ for $0 \leq k < n$.

(b) $soc({}_RR) = Rc^{n-1}$ is simple and essential in ${}_RR$.

© $Rc^k = J^k$ for $0 \leq k \leq n$.

Proof:

$(1) \Rightarrow (2)$. If $J^n = 0$ but $J^{n-1} \neq 0, n \geq 1$, let $0 \neq b \in J^{n-1}$. Then

$J \subseteq 1(b) \neq R$ so $J = 1(b)$ because R is local. Since R is left morphic,

$1(b) = Rc$ for some $c \in R$, and $c^n \in J^n = 0$.

$(2) \Rightarrow (3)$. Choose c as in (2) and assume that $c^{n-1} \neq 0$. Observe first that if $c^k \neq 0$ then $Rc^k \supset Rc^{k+1}$. For if $c^k = rc^{k+1}$, $r \in R$, then

$(1 - rc)c^k = 0$, whence $c^k = 0$ because $c \in J$. Hence

$R \supset Rc \supset Rc^2 \supset \cdots \supset Rc^n = 0, \ c^{n-1} \neq 0.$

Claim 1. $Rc^k - Rc^{k+1} = Uc^k$ for $0 \leq k < n$.

Proof.

Let x be an element of $Rc^k - Rc^{k+1}$, say $x = uc^k$, $u \in R$. Then $u \notin Rc = J$ because $x \notin Rc^{k+1}$, so u is a unit because R is local. Hence $x \in Uc^k$. Conversely, if $x = uc^k$, $u \in U$, then $x \notin Rc^{k+1}$ because otherwise we would have $c^k = u^{-1}x \in Rc^{k+1}$. This proves Claim 1.

Now let $L \neq 0$ be a left ideal of R. Since $L \subseteq Rc^0 = R$ and $L \nsubseteq Rc^n = 0$, there exists $k = 0,1, \ldots, n - 1$ such that $L \subseteq Rc^k$ and $L \nsubseteq Rc^{k+1}$.If

$x \in L - Rc^{k+1}$ then $x = uc^k$, $u \in U$, by Claim 1, so $c^k = u^{-1} \in L$.

Hence $L = Rc^k$.

$(3) \Rightarrow (4)$. This is clear.

$(4) \Rightarrow (5)$. If $R \supset L_1 \supset L_2 \supset \cdots L_m = 0$ is the lattice of left ideals of R, then R is local and $J = L_1 = Rc$ where c is any element of $L_1 - L_2$. Hence

$c^n = 0$ for some n because R is left artinian. If $r \in R$, we must show that

$r = uc^k$ with $u \in U$ and $k \geq 0$. This is clear if $r = 0$ or $r \notin J$ (because R is local). If $0 \neq r \in J = Rc$, write $r = s_1c$. If $s_1 \in U$ we are done; otherwise $s_1 \in J$ and we obtain $r = s_2c^2$. Continuing in this way completes the proof because c is nilpotent.

$(5) \Rightarrow (1)$. Choose c as in (5). Then $J \subseteq Rc$ because $R - Rc$ consistsof units by (5); we claim that $J = Rc$, that is $c \in J$. Indeed, if $r \in R$ then

$1 - rc \notin Rc$ by (5) because c is not a unit, so $1 - rc$ is a unit for every $r \in R$. Hence $J = Rc$, and so R is local by (5).

Claim 2.

$1(c^k) = Rc^{n-k}$ for $0 \leq k < n$.

Proof.

It is clear if $k = 0, n$. We have $Rc^{n-k} \subseteq 1(c^k)$ because $c^n = 0$. Conversely, let $x \in 1(c^k)$. Since the conditions in (2) are satisfied for R, Claim 1 gives

$x = uc^m, u \in U, m \leq n$. Then $0 = xc^k = uc^{m+k}$ so $c^{m+k} = 0$. This means that $m + k \geq n$, so $m \geq n - k$, whence $x = uc^m \in Rc^{n-k}$.

This proves Claim 2.

Now suppose that $a \in R$, say $a = uc^k, u \in U, k \geq 0$. Then

$1(a) \cong 1(a)u = 1(c^k) = Rc^{n-k}$ by Claim 2; and $Ra = Ruc^k = Rc^k = 1(c^{n-k})$, again by Claim 2. This proves (1).

Finally, (a) follows from Claims 1 and 2, and (b) follows from (3). We prove (c) by induction on k, the case $k = 0, 1$ being clear by (2). If

$J^k = Rc^k$ for some k, then $J^{k+1} = J.Rc^k \subseteq J.c^k = Rc.c^k = Rc^{k+1}$. Since $c \in J$, this proves ©.

For convenience , we refer to the rings in Theorem 9 as left special rings. Note that the left special rings with $J = 0$ are just the division rings.

Corollary 10.

Let R be left special with $J = Rc$ as in Theorem $9(2)$. If R is also right special, then $J = cR$ (and so the left-right analogues of the properties in Theorem 9 hold).

Proof.

The case $J = 0$ is clear,so assume $J \neq 0$ and let $J = bR$. Then $b \in Rc$ and $b \notin J^2 = Rc^2$ because $J^2 \neq J$ by Theorem 9. Hence $b = uc$ where u is a unit of R (again by Theorem 9), and so $cR = u^{-1}bR = u^{-1}J = J$.

Example 11.

The ring R in Example 8 is left special but not right special.

Proof.

If R were right special then (using the notation of Example 8) we would have $J = cR = \bar{F}c$ by Corollary 10, and hence would contradict the fact that $\bar{F} \neq F$.

Note that every left special ring R is a left duo ring (that is every left ideal is a right ideal). However if F is a field then $M_2(F)$ is a left and right morphic ring (it is unit regular), but is neither left nor right duo.

If p is a prime , the ring $\mathbb{Z}_{pn}$ is left and right special for every $n \geq 1$ by

Theorem 9. Hence Example 2 gives:

Example 12.

The ring $\mathbb{Z}_n$ of integers modulo n is left and right morphic for every

$n \geq 2$.

Note that every proper image of the ring $\mathbb{Z}$ of integers is morphic , but $\mathbb{Z}$ itself is not morphic. In fact , a similar argument shows that every proper image of any commutative principal ideal domain is morphic.

A ring R is said to be left Kasch if every simple left R-module embeds in

${}_R R$, equivalently if $r(L) \neq 0$ for every (maximal) left ideal L of R. In a left morphic ring this condition has profound implications for the maximal left ideals.

Proposition 13.

The following are equivalent for a left morphic ring R:

(1) R is left Kasch.

(2) Every maximal left ideal of R is an annihilator.

(3) Every maximal left ideal of R is principal.

Proof.

(1) $\Rightarrow$ (2) holds without the left morphic hypothesis.

(2) $\Rightarrow$ (3). If L is a maximal left ideal of R, let $L = 1(X)$ by (2) where X is a nonempty subset of R. If $0 \neq a \in X$, then $L \subseteq 1(a) \neq R$ so, again ,

$L = 1(a)$ by maximality. Hence $L = Rb$ for some $b \in R$ because R is

left morphic.

$(3) \Rightarrow (1).$ If L is a maximal left ideal of R, let $\mathrm{L} = \mathrm{Rb}, \mathrm{b} \in \mathrm{R}$, by (3). By the left morphic hypothesis let $\mathrm{Rb} = 1(\mathrm{a})$ for some $\mathrm{a} \in \mathrm{R}$. Then

$0 \neq \mathrm{a} \in \mathrm{r}(\mathrm{L})$, and (1) follows.

Question.

If R is a left morphic, left Kasch ring, is R right Kasch?

2. Corners and matrix rings

We are going to prove that if R is left morphic the same is true of the corner ring eRe for any $e^2 = e \in R$. The following lemma stems from a result of Lam and Murray [9] in the unit regular case.

Lemma 14.

Let $e^2 = e \in \mathrm{R}$ and write $\mathrm{f} = 1 - \mathrm{e}$. The following conditions are equivalent for $\mathrm{a} \in \mathrm{eRe}$.

(1) a is left morphic in eRe.

(2) $\mathrm{a} + \mathrm{b}$ is left morphic in R for all left morphic elements b in fRf.

(3) $\mathrm{a} + \mathrm{b}$ is left morphic in R for all units b in fRf.

(4) $\mathrm{a} + \mathrm{f}$ is left morphic in R.

(5) $\mathrm{a} + \mathrm{b}$ is left morphic in R for some unit b in fRf.

Proof.

First, $(1) \Rightarrow (2)$ follows by Example 2 if we view $\mathrm{a} + \mathrm{b}$ as in $\mathrm{eRe} \times \mathrm{fRf}$, and $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ is clear. Given b as in (5), choose $\mathrm{c} \in \mathrm{R}$ such that $1(\mathrm{a} + \mathrm{b}) = \mathrm{Rc}$ and $1(\mathrm{c}) = \mathrm{R}(\mathrm{a} + \mathrm{b})$. We show that $\mathrm{c} \in \mathrm{eRe}$,

$1_{\mathrm{eRe}}(\mathrm{a}) = (\mathrm{eRe})\mathrm{c}$, and $1_{\mathrm{eRe}}(\mathrm{c}) = (\mathrm{eRe})\mathrm{c}$.

To see that $\mathrm{c} \in \mathrm{eRe}$, note first that $0 = (\mathrm{a} + \mathrm{b})\mathrm{c} = \mathrm{ac} + \mathrm{bc} \in \mathrm{eR} \oplus \mathrm{fR}$, so $\mathrm{bc} = 0$. As b is a unit in fRf, it follows that $\mathrm{fc} = 0$, that is

$\mathrm{ec} = \mathrm{c}$. Similarly $\mathrm{ce} = \mathrm{c}$, so $\mathrm{c} \in \mathrm{eRe}$.

Next $1_{\mathrm{eRe}}(\mathrm{a}) = (\mathrm{eRe})\mathrm{c}$. Indeed, let $\mathrm{xa} = 0$ where $\mathrm{x} \in \mathrm{eRe}$. Then

$\mathrm{x} \in 1(\mathrm{a} + \mathrm{b}) = \mathrm{Rc}$ so $\mathrm{x} \in \mathrm{eRe} \cap \mathrm{Rc} = (\mathrm{eRe})\mathrm{c}$ by hypothesis

(since $\mathrm{c} \in \mathrm{eRe}$). Conversely, let $\mathrm{x} \in (\mathrm{eRe})\mathrm{c}$. Then

$\mathrm{x} \in \mathrm{Rc} = 1(\mathrm{a} + \mathrm{b})$, so $0 = \mathrm{x}(\mathrm{a} + \mathrm{b}) = \mathrm{xa}$ and we have $\mathrm{x} \in 1_{\mathrm{eRe}}(\mathrm{a})$.

Finally, we have $(eRe)a = (eRe)(a + b) \subseteq R(a + b) = 1_R(c)$, and it follows that $(eRe)a \subseteq 1_{eRe}(c)$. Conversely, if $x \in 1_{eRe}(c)$ then

$x \in 1_R(c) = R(a + b)$, so $x = exe \in (eRe)a$.

This proves that $1_{eRe}(c) = (eRe)a$.

Note that Lam and Murray [9] construct a regular ring R and an element $a \in eRe$, $e^2 = e \in R$, such that a is unit regular in R but not in

eRe. Hence a is left morphic in R but not in eRe (by Erlich's theorem because eRe is regular – see proposition 5). However, the left morphic condition passes from a ring R to any corner of R.

Theorem 15.

If R is a left morphic ring the same is true of eRe for every idempotent

$e \in R$.

Proof. Write $f = 1 - e$. If $a \in eRe$ then $a + f$ is left morphic by hypothesis, so the result follows from Lemma 14.

Any hope that 'left morphic' is a Morita invariant is dashed by the following example.

Example 16.

If R is the ring of Example 8 then R is left morphic but $M_2(R)$ is not left morphic.

Proof.

We use the notation of Example 8, where it is shown that R is left morphic . If $y \in F - \bar{F}$, we show that

$$\lambda = \begin{bmatrix} c & yc \\ 0 & 0 \end{bmatrix}$$

is not left morphic in $S = M_2(R)$. Since $1_R(c) = J = Fc$, we have

$$1_S(\lambda) = \begin{bmatrix} Fc & R \\ Fc & R \end{bmatrix}.$$

Suppose there exists $\mu \in S$ such that $S\lambda = 1_S(\mu)$ and $S\mu = 1_S(\lambda)$. Then μ has the form

$$\mu = \begin{bmatrix} xc & r \\ zc & s \end{bmatrix},$$

and the condition $\lambda\mu = 0$ implies that $cr + ycs = 0$. If we write

$r = z_1 + w_1 c$ and $s = z_2 + w_2 c$ in R, then $cz_1 + ycz_2 = 0$, whence

$\bar{z}_1+y\bar{z}_2=0$, a contradiction if $\bar{z}_2\neq 0$ because $y\notin\bar{F}$. So $\bar{z}_2=0$, whence

$\bar{z}_1=0$ and so $z_1=0=z_2$. It follows that

$$\mu=\begin{bmatrix} xc & w1c \\ zc & w2c \end{bmatrix}$$

is in $M_2(J)$, and hence that

$$\begin{bmatrix} Fc & R \\ Fc & R \end{bmatrix}=1_S(\lambda)=S\mu\subseteq M_2(J),$$

a contradiction.

The next proposition identifies one situation where $M_n(R)$ is left and right morphic.

Theorem 17.

Let R be a left and right special ring. Then $M_n(R)$ is left and right morphic for each $n\geq 1$.

Proof.

Let $\lambda\in M_n(R)$. Observe that if $\lambda\to\mu$ in $M_n(R)$ by row or column operations then (by Lemma 3) λ is left morphic if and only if μ is left morphic. Since R is left and right special, let $J=J(R)=Rc=cR$ where

$c^m=0$ (as in Corollary 10). Hence λ has the form $\lambda=[u_{ij}c^{k_{ij}}]$ where, for each i and j, u_{ij} is a unit and $0\leq k_{ij}\leq m$. Using row and column operations we may assume that $c^{k_{11}}$ is the smallest power of c appearing in row 1 or column 1 of λ. Then, again using row and column operations, we may assume that λ has the form

$$\lambda=\begin{bmatrix} ck11 & 0 \\ 0 & \mu \end{bmatrix}$$

where $\mu\in M_{n-1}(R)$. Continuing we may assume that $\lambda=\mathrm{diag}(c^{k_{11}},\ldots c^{k_{nm}})$ is a diagonal matrix. Since each $c^{k_{ii}}$ is left morphic it follows that λ is left morphic .

Question.

If R is left and right morphic, is the same true of $M_2(R)$? Equivalently (with Theorem 15), is ‘left and right morpohic’ a Morita invariant?

This is true if R is unit regular [6, Corollary 3], but see Example 16.

The next result gives insight into when a matrix ring is left morphic. Recall that a Morita context is a four-tuple (R,V,W,S) where $V={}_RV_S$ and $W={}_SW_R$ are bimodules and there exist multiplications

$V \times W \to R$ and $W \times V \to S$ such that

$$C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$$

is an associative ring with the usual matrix operations (called the context ring).

Proposition 18.

Let $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a context ring and assume that C is left morphic. If either $VW \subseteq J(R)$ or $WV \subseteq J(S)$, then $V = 0$ and $W = 0$.

Proof.

Assume that $WV \subseteq J(S)$; the argument is similar if $VW \subseteq J(R)$. Let

$v \in V$, and write $\lambda = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$ in C. Then

$$1_C(\lambda) = \begin{bmatrix} 1R(v) & V \\ 1W(v) & S \end{bmatrix}.$$

Let $\mu = \begin{bmatrix} a & v0 \\ w0 & b \end{bmatrix} \in C$ be such that $1_C(\lambda) = C\mu$ and $1_C(\mu) = C\lambda$. Then

$\lambda\mu = 0$ implies $vb = 0$, and $C\mu = 1_C(\lambda)$ gives $Wv_0 + Sb = S$. In particular , $WV + Sb = S$ so $Sb = S$ because $WV \subseteq J(S)$. It follows that b is a unit because S is left morphic by Theorem 15. Hence $v = 0$

(because $vb = 0$) and so $V = 0$. Thus $VW = 0$ so, similarity, $W = 0$.

Corollary 19.

Let e and f be orthogonal idempotents in a left morphic ring R. If

$eRf \subseteq J$ then $eRf = 0 = fRe$.

Proof.

We have the pierce decomposition $(e + f)R(e + f) \cong \begin{bmatrix} eRe & eRf \\ fRe & fRf \end{bmatrix}$ so Theorem 15 and proposition 18 apply.

If $e \in R$ is an idemptent and $eR(1 - e) = 0$, then $[(1 - e)ReR]^2 = 0$. Hence, if R is semiprime (in particular, unit regular),we have

$[(1 - e)Re = 0$ and e is central. This holds in any left morphic ring by Corollary 19:

Corollary 20.

If e is an idempotent in a left morphic ring R, then e is central if and only if $eR(1-e)=0$ (equivalently $(1-e)Re=0$).

We can remove the restriction that e and f are orthogonal in Corollary 19 if $eRf=0$.

Theorem 21.

If e and f are idempotents in the left morphic ring R, then $eRf=0$ if and only if $fRe=0$.

Proof.

Assume that $eRf=0$, and write $h=f-fe$. Since $ef=0$, h is an idempotent orthogonal to e, and $eRh=0$. Hence $hRe=0$ by Corollary 19, so $fre=fere$ for each $r\in R$. In particular, if $a\in fRe$ we have

$a=fae=feae=0$ because $ea\in e(fRe)=0$.

We are going to apply proposition 18 to idempotents in a left morphic ring, and the next result will be needed. An idempotent $e^2=e\in R$ is called full (in R) if $ReR=R$.

Lemma 22.

If $e^2=e$ in a ring R, then $1-e$ is full in R if and only if

$eR(1-e)Re=eRe$.

Proof.

For convenience , write $\bar{e}=1-e$. If $r\in R$, the fact that

$r=ere+er\bar{e}+\bar{e}re+\bar{e}r\bar{e}$ shows that $R=eRe+R\bar{e}R$ as $\mathbb{Z}$-modules. If $eR\bar{e}Re=eRe$ this gives $R=R\bar{e}R$ so $\bar{e}$ is full in R.

The converse is clear.

An idempotent e in ring R is called local if eRe is a local ring.

Theorem 23.

Let R be a left morphic ring.

(1) If $e^2=e\in R$ is local , then $1-e$ is either full or central.

(2) If e and f are orthogonal local idempotents in R then $eRf\neq 0$ if and only if $eR\cong fR$.

Proof.

(1) Write $\bar{e} = 1 - e$. If $\bar{e}$ is not full then $eR\bar{e}Re \neq eRe$ by Lemma 22, so

$eR\bar{e}Re \subseteq J(eRe)$ because eRe is local. But then applying proposition 18 to the context ring

$$C = \begin{bmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{bmatrix}$$

gives $eR\bar{e} = 0 = \bar{e}Re$, so e is central and (1) follows.

(2) If $eRf \neq 0$ then $eRf \nsubseteq J$ by Corollary 19, so choose

$0 \neq a \in eRf - J$ and define $\lambda: fR \to eR$ by $\lambda(x) = ax$. Then

$im(\lambda) = aR \nsubseteq eJ$, so λ is epic because e is local. But then λ splits because eR is projective , and so λ is monic because fR is indecomposable. Hence λ is an isomorphism , so $eR \cong fR$. Since the converse is clear, this proves (2).

Observe that the only property of the local idempotent e used in (1) of Theorem 23 is that $J(eRe) = eJe$ is the unique maximal two sided ideal of eRe.

3. P-injectivity

A ring R is called right P-injective (more formally, right principally injective) if, for every $a \in R$, every R-linear map $\gamma: aR \to R_R$ can be extended to $R_R \to R_R$,that is $\gamma = c$. Is left multiplication by some $c \in R$. Hence every regular ring is both right and left P-injective. It is a routine matter to verify (see [11, Lemma 1.1]) that R is right P-injective if and only if $1r(a) = Ra$ for each a. But if R is left morphic, and if $Ra = 1(b)$ for some $b \in R$, then $1r(a) = 1r(Ra) = 1r1(b) = 1(b) = Ra$. This proves (1) in the following result.

Theorem 24.

Let R be a left morphic ring. Then:

(1) R is ring P-injective.

(2) $Z(R_R) = J$.

(3) $soc(R_R) \subseteq soc({}_RR)$.

(4) If kR is simple, $k \in R$, then Rk is simple.

Proof.

(1) is provd above, (2) is by [11, Theorem2.1], and (3) and (4) are by

[12 , Theorem 1.14].

Example 25.

There exists a right and left P-injective ring that is neither left nor right morphic.

Proof.

Let R be a regular ring that is not unit regular, for example $R = \mathrm{end}(V)$ where V is a vector space of countably infinite dimension over a field. Then R is right (and left) P-injective because every principal right (left) ideal is a direct summand, but it is not left (or right) morphic because it is not directly finite.

Example 26.

The ring in Example 8 is right P-injective but not right morphic.

Proof.

R is left morphic by Lemma 7, and hence right P-injective (by Theorem 24). However , R is not left P-injective (indeed if $y \in F - \bar{F}$, the map

$Fc \to_R R$ given by $xc \mapsto xyc$ does not extend to ${}_R R \to {}_R R$). Hence R is not right morphic (by Theorem 24).

Proposition 27.

Let R be a left morphic ring (hence right P-injective). Then R is left P-injective if and omly if it is right morphic.

Proof.

Assume that R is left P-injective , so $r1(a) = aR$ for each $a \in R$. Given $a \in R$, choose $b \in R$ such that $Ra = 1(b)$ and $Rb = 1(a)$. Then

$r(a) = r(Ra) = r1(b) = bR$ by the left P-injective hypothesis. Similarly, $aR = r(b)$, so R is right morphic.

The converse is by Theorem 24.

In a left morphic ring R we know that $Z(R_R) = J$ by Theorem 24. We conclude this section with some observations concerning the left singular ideal $Z({}_R R)$. Recall that a ring R is said to be reduced if it has no nonzero nilpotent elements.

Proposition 28.

Let R be a left morphic ring.

(1) $Z({}_R R) \subseteq J$.

(2) If R is reduced then $Z({}_RR) = 0$ and R is a left duo ring.

(3) The following are equivalent:

(a) ${}_RR$ is uniform.

(b) $Z({}_RR)$ is the set of nonunits.

(c) R is local and $Z({}_RR) = J$.

Proof.

If $a \in R$, let $b \in R$ satisfy $Ra = 1(b)$ and $Rb = 1(a)$.

(1) If $a \in Z({}_RR)$ then $1(a) \subseteq^{ess} {}_RR$, so $1(1 - a) = 0$

(because $1(a) \cap 1(1 - a) = 0$). Hence $1 - a$ is a unit by proposition 6 so $a \in J$.

(2) If a and b are as above, we have $(Ra \cap Rb)^2 \subseteq Ra.1(a) = 0$ because $xa = 0$ implies $ax = 0$, so $Ra \cap Rb = 0$. Hence

$Ra \cap 1(a) = 0$ so, if $a \in Z({}_RR)$ then $Ra = 0$. Hence $Z({}_RR) = 0$. For the rest, it suffices to show that $aR \subseteq Ra$. If $r \in R$ then $b(aR) = 0$ so

$(ar)b = 0$ by hypothesis. Hence $ar \in 1(b) = Ra$, as required.

(3) Given (a), let a be a nonunit and write $1(a) = Rb, b \in R$. Then

$b \neq 0$ by proposition 6, so $1(a) = Rb \subseteq^{ess} {}_RR$ by (a), and it follows that

$a \in Z({}_RR)$. This proves (b). Given (b) the set of nonunits is an ideal

$Z({}_RR)$, necessarily equal to J, and © follows. Finally, assume that © holds . If $0 \neq a \in R$ and $Ra = 1(b), b \in R$, then b is a nonunit (as

$a \neq 0$) so $b \in Z({}_RR)$ by ©. Hence $Ra = 1(b) \subseteq^{ess} {}_RR$, proving (a).

Note that the ring R in Example 8 is left morphic and satisfies

$Z({}_RR) = J$ but R is not right morphic. This ring also has the property that

$Z({}_RR) = J = Z(R_R)$ but R_R is not uniform.

Question.

If R is a semiperfect ,left morphic ring, is $J(R) = 0$?

4. Structure theorems

If a mild finiteness condition is applied to a left morphic ring, we can obtain some structure results. To begin, Therefore 23 leads to the following theorem for semiperfect, left morphic rings.

Theorem 29.

A ring R is semiperfect and left morphic if and only if

$R \cong M_{n1}(R_1) \times M_{n2}(R_2) \times \ldots . \times M_{nk}(R_k)$,

where each $M_{ni}(R_i)$ is left morphic and $R_i \cong e_iRe_I$ for some local idempotent $e_i \in R$.

Proof.

Let E denote a finite set of orthogonal , local idempotents in R whose sum is 1. Given e and f in E, define $e \sim f$ if and only if $eRf \neq 0$. This is an equivalence relation on E by Theorem 23, so let $E_1, \ldots E_m$ denote the equivalence classes and write $h_k = \sum\{e \in E | e \in E_k\}$ for each k. Then

$h_kRh_k \cong M_{nk}(eRe)$ for any $e \in E_k$ using Theorem 23, and h_kRh_k

(and eRe) is left morphic by Theorem 15. Hence it remains to prove that each h_k is central. But if $h \neq l$ then $h_kRh_l \subseteq \sum\{eRf | e \in E_k, f \in E_l\} = 0$ because $e \nsim f$. Hence, if $r \in R$, $h_kr = h_kr(h_1 + \ldots . + h_m) = h_krh_k$. Similarly , $rh_k = h_krh_k$ and it follows that h_k is central.

We hasten to note that $M_n(R)$ need not be left morphic even if R is left special as Example 16 shows. However we do get a better theorem in the semiprimary case.

Corollary 30.

The following are equivalent for a ring R:

(1) R is a semiprimary ring that is left and right morphic.

(2) $R \cong M_{n1}(R_1) \times M_{n2}(R_2) \times \ldots \ldots \times M_{nk}(R_k)$ where each R_i is left and right special.

Proof.

(1) ⇒ (2). Assume that (1) holds. If e is a local idempotent in R, then eRe is left and right morphic by Theorem 15, and $J(eRe) = eJe$ is nilpotent. Hence eRe is left and right special, and (2) follows from Theorem 29.

(2) ⇒ (1). Given the situation in (2), each $M_{ni}(R_i)$ is left and right morphic by Theorem 17, and $J(M_{ni}(R_i)) = M_{ni}(J(R_i))$ is nilpotent because $J(R_i)$ is nilpotent . Hence (1) follows from Example 2.

In fact, the rings in Corollary 30 are all left and right artinian (this is true of left and right special rings), and we present several characterizations of these rings below (Theorem 35). This entails an examination of the effect on a left morphic ring of various finiteness conditions. We begin with the ascending chain condition on right annihilators.

Theorem 31.

Let R be a left morphic ring with ACC on right annihilators.Then:

(1) eRe is left special for every local idempotent $e \in R$.

(2)R is left artinian.

(3) R is right and left Kasch.

(4) $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$.

(5) $Z({}_RR) = J = Z(R_R)$.

Proof.

We have $J = Z(R_R)$ because R is right P-injective (Theorem 24), so J is nilpotent by the ACC on right annihilators (this is the Mewborn-Winton theorem [10]). Hence (1) follows from Theorems 15 and 9. Moreover, R satisfies the DCC on left annihilators, and so has the DCC on principal left ideals because it is left morphic. This means that R is right perfect by Bass' theorem [1] (see [8, Theorem 23.20]). Now (2) follows from (1) and Theorem 29. Finally, R is a semiperfect, right P-injective ring in which $\mathrm{soc}(R_R) \subseteq^{ess} R_R$ (because J is nilpotent), and so R is a right GPF ring as defined in [11, p.83]. Hence (3)follows by [11, Corollary 2.3], (4) by [11,Theorem 2.3] and (5) by [11, Corollary 2.2].

Note that every left special ring is left duo and satisfies the ACC on right annihilators (it is left artinian). Hence Theorem 9 gives:

Corollary 32.

A left duo, left morphic ring has ACC on right annihilators if and only if it is a finite direct product of special left morphic rings.

The converse to Theorem 31 is not true. In fact if R is the ring in Example 8 then $M_2(R)$ enjoys properties (1)-(5) in Theorem 31 but it is not left morphic by Example 16. However we do have Theorem 35 below, but the proof requires the following lemma.

Lemma 33.

The following are equivalent for a semiperfect, left morphic ring R:

(1) J is nilpotent.

(2) J is nil and $soc(R_R) \subseteq^{ess} R_R$.

(3) R has ACC on principal left ideals and $soc(R_R) \subseteq R_R$.

Proof.

$(1) \Rightarrow (2)$ and (3). Given (1), then (2) is clear, and R is left perfect and so has DCC on principal right ideals by Bass' theorem [1] (see [8, Theorem 28.20]). Hence R has ACC on principal left ideals by Jonah's theorem [7].

(2) or $(3) \Rightarrow (1)$. Let $1 = e_1 + \cdots + e_n$ be orthogonal local idempotents in R. Then $J = Je_1 + \cdots + Je_n$, so it suffices to show that each Je_i is a nilpotent left ideal. But $(Je_i)^{k+1} = J(e_iJe_i)^k$ for each $k \geq 0$, so it suffices to show $e_iJe_i = J(e_iRe_i)$ is nilpotent.Now observe that each e_iRe_i is local and left morphic, and that either $J(e_iRe_i) = e_iJe_i$ is nil or e_iRe_i inherits the ACC on principal left ideals (by a routine argument). Moreover,e_iRe_i has essential right socle by Theorem 29 because this property passes to direct factors and is a Morita invariant [12, Lemma 3.17]. Hence we may (and do) assume that R is local.

Since $soc(R_R) \subseteq soc({}_RR)$ by Theorem 24, we have $soc({}_RR) \neq 0$ by hypothesis . So let Rk be a simple left ideal of R , and choose $c \in R$ such that $Rk = 1(c)$ and $Rc = 1(k)$. Then $R/Rc= \frac{R}{1(k)} \cong Rk$, so Rc is a maximal right ideal of R. Hence $Rc = J$ because R is local. Then the proof of © in Theorem 9 goes through to show that $J^k= Rc^k$ for all

$k \geq 0$, so we are done if J is nil. On the other hand , assume that R has the ACC on principal left ideals. Then the chain $1(c) \subseteq 1(c^2) \subseteq 1(c^3) \subseteq$

Terminates (it consists of principal left ideals because R is left morphic) , say $1(c^m) = 1(c^{m+1})$ where $m \geq 1$. Choose $x, y \in R$ such that

$Rx = 1(c^m), 1(x) = Rc^m = J^m$, and $Ry = 1(c^{m+1}), 1(y) = Rc^{m+1} = J^{m+1}$.

Then $Rx = Ry$, say $x = uy$ and $y = vx$. Hence $x = uvx$, so $uv \notin J$ because $x \neq 0$. It follows that uv is a unit (as R is local) and so u is a unit (by proposition 6). But $0 = c^m x = c^m uy$, so $c^m u \in 1(y) = Rc^{m+1} = J^{m+1}$. Thus $c^m \in J^{m+1} = Rc^{m+1}$, say $c^m = rc^{m+1}$. Hence $(1 - rc)c^m = 0$ so

$c^m = 0$ because $c \in J$. This means $J^m = Rc^m = 0$, as required.

Note that the proof of (2) or $(3) \Rightarrow (1)$ in Lemma 33 requires only that

$soc({}_RR) \subseteq^{ess} R_R$. What is needed is that the condition $soc({}_RR) \neq 0$ passes from R to eRe for each local idempotent $e \in R$.

We can now prove a structure theorem for left perfect, left and right morphic rings. Recall that a ring R is called right selfinjective if every R-linear map $\gamma: T \to R_R$, T a right ideal of R, extends to $R_R \rightarrow R_R$, equivalently if $\gamma = c$ is left multiplication by some $c \in R$. A left and right selfinjective ring R is called quasi-Frobenius if it is left and right artinian.

Lemma 34.

Every left and right special ring R is quasi-Frobenius.

Proof.

R is left and right artinian by Theorem 9. Since every one-sided ideal is principal, R is right and left selfinjective by Theorem 24.

Theorem 35.

The following are equivalent for a ring R:

(1) R is left artinian and left and right morphic.

(2) R is semiprimary and left and right morphic.

(3) R is left perfect and left and right morphic.

(4) R is a semiperfect, left and right morphic ring in which J is nil and

$\mathrm{soc}(R_R) \subseteq^{ess} R_R$.

(5) R is a semiperfect, left and right morphic ring with ACC on principal left ideals in which $\mathrm{soc}(R_R) \subseteq^{ess} R_R$.

(6) R is a finite direct product of matrix rings over right and left special rings.In this case, R is quasi-Frobenius.

Proof.

(1) ⇒ (2) ⇒ (3) is clear; and (6) ⇒ (1) because such a direct product is left artinian by Theorem 9, and it is left and right morphic by

Theorem 17.

(3) ⇒ (6) and (5) ⇒ (6). By Theorem 29, $R \cong \prod_{i=1}^{m} M_{n_i}(R_i)$ where each $R_i \cong e_iRe_i$ for some local idempotent $e_i \in R$. Hence each R_i is local and left and right morphic by Theorem 15. Moreover R_i is left special by Theorem 9 because $J(R_i)$ is nilpotent for each i by Lemma 33.

This proves (6).

Finally, R is quasi-Frobenius by (6) and Lemma 34 because being quasi-Frobenius is a Morita invariant.

The converse to Lemma 34 is false.

Example 36.

If C_2 denotes the group of order 2, the group ring $R = \mathbb{Z}_4C_2$ is a commutative, local quasi-Frobenius ring which is not morphic.

Proof.

First, R is selfinjective by Connell's theorem [4, Theorem 4.1] so, since it is clearly artinian, R is quasi-Frobenius. Writing $C_2 = \{1, g\}$, we have ideals $A = \{a + bg | a + b = 0\}$ and $B = \{a + bg | a - b = 0\}$ in R, and a routine calculation shows that every element $u \notin A + B$ satisfies

$u^2 = 1$. Hence R is local and $A + B = J$. But $A \nsubseteq B$ and $B \nsubseteq A$, so R is not morphic by Theorem 9. However the only element of R that is not morphic is $x = 2 + 2g$, as the reader can verify.

Observe that if F is a field, the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is left artinian and $eRe \cong F$ is left special for each local idempotent e, but R is neither left nor right morphic (indeed, neither left nor right P-injective). In fact, neither $soc\,(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ or $soc({}_RR) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ contains the other.

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