NUMERICAL STUDY OF HEAT TRANSFERS IN TISSUES DURING HYPERTHERMIA USING MODIFIED BERNSTEIN POLYNOMIALS

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Abstract

In the present article, we use modified Bernstein polynomial (B–polynomial) as a basis for the numerical approximation of heat transfer in hyperthermia treatment. A set of continuous polynomials over the spatial domain is use to expand the desired solution using discretization in time variable only. The Galerkin method is use to determine the expansion coefficients to construct initial trial functions. The system of equations has been solved using fourth–order Runge-Kutta method. The accuracy of the solutions is dependent on the size of the B–polynomial basis set. Also, Homotopy Perturbation Method has been applied to solve Matrix form of initial value differential equations which is transformed from boundary value differential equation of desired problem by using central difference scheme. The results thus obtained are in very good agreement with the previous results and it is presented graphically.

Keywords: *Bio-heat equation; Bernstein polynomial; Galerkin method; Finite difference equations; Homotopic perturbation method*

1. Introduction:

In the hyperthermia treatment, the temperature of the tumor is elevated to significantly higher values than the surrounding tissue. The success of this type of treatment is substantially less in tumors than normal tissue or that the electric permittivity of cancerous cells is different from normal cells, resulting in enhanced E.M. absorption in the tumor. The primary rational for the therapeutic heating is the inducement of blood flow changes which are expected to be associated with the increased supply of nurturing substances (e.g. oxygen, leukocytes, nutrients etc.) to the damaged tissue as well as the removal of toxic waste and cellular debris.

The most common model, called bio-heat equation, first proposed by Pennes [1], which contains a separate blood perfusion term. The bio-heat equation is a relatively simple model. This fact represents both its greatest advantage, mathematical tractability, and its greatest shortcoming, over simplification to a scalar form of an effect which has definite vectoral components. A more promising model was proposed by Chen & Holmes [2], who suggested two additional terms in the bio-heat equation: one proportional to the local blood perfusion velocity and temperature gradient and the other one representing an effective thermal conductivity due to capillary perfusion.

New bio-heat equation is proposed by Weinbaum and Jiji [3] on the basis of their observation that in the muscle near the skin surface the thermally significant arteries and veins run next to each other in a parallel, counter flow pattern. Also, Kouremenos and Antonopoulous [4] discussed the heat transfer in tissues radiated by a 432 MHz. directional antenna by using finite difference technique on bio-heat equation. K. N. Rai & S. K. Rai [5] have given their numerical study on heat transfer in tissues during Hyperthermia using bio-heat equation.

In this paper we have considered a cylindrical model in which a single blood vessel of radius R_1 is surrounded by a tissue of radius R_2. Our interest is to find out the dependency of temperature distribution over different physical parameters of blood vessel and surrounding tissue, for this we present a numerical approach to solving above problem which requires discretization of the time variable t only. B-polynomials [6, 7 & 8] are used as trial functions and Galerkin method is used with initial data to obtain a system of equations in spatial variable *x*. The system of equations has been integrated using a smaller time step in a fourth-order Runge-Kutta scheme. In the following sections we provide details of the numerical method to solve the problem. We also present our result graphically and compare with another numerical result which is getting by Homotopic Perturbation Method [9 & 10] applying on initial valued matrix form of the problem.

2. Description and Formulation of the Problem:

The geometry of the angularly symmetric model is given in Fig. (1), Initially, the tissue is at constant temperature T0 (T0 =37°C). Consider, the tissue is heated by electromagnetic radiation using a 432 MHz. antenna in such a manner that whole periphery is heated simultaneously [4]. During heating process, the surface of the tissue is always maintained at a temperature Ts by some artificial means. Under the above assumptions the differential equation governing the process of heat transfer in the tissue may be written as:

$$\rho C \frac{\partial T}{\partial t} = \frac{k_t}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + Q_R + Q_m + Q_b \qquad \text{...... (1)}$$

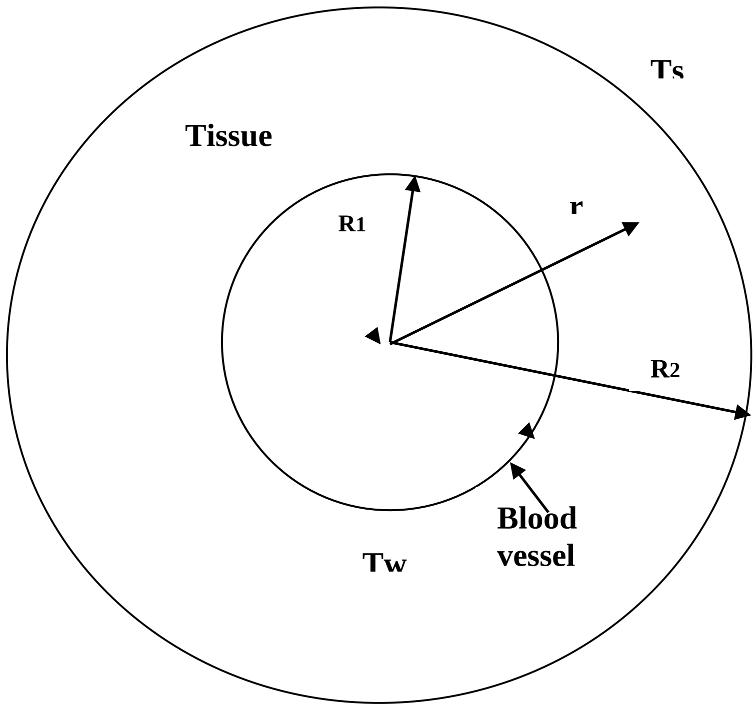

Fig. 1. Geometry of the Problem

$$Q_r = \rho \mathrm{SP} \exp[a(R_2 - r - 0.01)] \quad \text{...... (2)}$$

According to the Pennes bio-heat transfer equation Qb may be expressed as

$$Q_b = W_b\, C_b\, (Ta - T) \quad \text{...... (3)}$$

Where, T_a stands for the arterial blood temperature which may be taken either constant or variable. In this paper it is taken as constant.

The term Q*m* in equation (1) is temperature dependent. Its dependence on temperature can be written as

$$Q_m = Q_{m0}\left[1 + 0.1(T - T_0)\right] \quad \text{...... (4)}$$

The initial and boundary conditions are as follows:

$$T(r,0) = T_0 \qquad \text{...... (5)}$$

$$T(R_1, t) = T_w \qquad \text{...... (6)}$$

$$T(R_2, t) = T_s \qquad \text{...... (7)}$$

B-polynomial basis function

We implement a new algorithm based on a modification to the type of bases for the space of polynomials, the Bernstein functions. The procedure takes advantage of the continuity and unity partition properties of the basis set of B-polynomials [6] over a closed interval. This provides greater flexibility in which to impose boundary conditions at the end points of the interval [R1, R2].

The general form of the B-polynomials of *n*th degree are define as

$$B_{i,n}(r) = \binom{n}{i} \frac{(r - R_1)^i (R_2 - r)^{n-i}}{(R_2 - R_1)^n} \qquad \text{......... (8)}$$

for i = 0, 1, ……, *n* where the binomial coefficients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \; . \qquad \text{......(9)}$$

Here $n! = 1 \times 2 \times 3 \times \cdots \times n$ or $n \geq 1$ and $0! = 1$. B-polynomials defined above form a complete basis over the interval [R_1, R_2]. There are (n + 1) polynomials with degree *n*. For convenience, we set $B_{i,n}(r) = 0$ if i < 0 or i > n. The details of these polynomials with their recurrence relations can also be found in the work of Bhatta and Bhatti [8]. It can easily be shown that each of the B-polynomials is positive and also the sum of all the B- polynomials is unity for all real $r \in [R_1, R_2]$, i.e., $\sum_{i=0}^{n} B_{i,n}(r) = 1$. It is easy to show that any given polynomial of degree n can be expressed in terms of linear combination of the basis functions. Recurrence formula and derivatives of B-polynomials are given by

$$B_{i,n}(r) = \left(\frac{R_2 - r}{R_2 - R_1}\right) B_{i,n-1}(r) + \left(\frac{r}{R_2 - R_1}\right) B_{i-1,n-1}(r) \qquad \ldots\ldots (10)$$

$$B'_{i,n}(r) = \frac{n}{R_2 - R_1}[B_{i-1,n-1}(r) - B_{i,n-1}(r)] \qquad \ldots\ldots (11)$$

$$B''_{i,n}(r) = \frac{n(n-1)}{(R_2 - R_1)^2}[B_{i-2,n-2}(r) - 2B_{i-1,n-1}(r) + B_{i,n-2}(r)] \qquad \ldots\ldots (12)$$

$$B'''_{i,n}(r) = \frac{n(n-1)(n-2)}{(R_2 - R_1)^3}[B_{i-3,n-3}(r) - 3B_{i-2,n-3}(r) + 3B_{i-1,n-3}(r) - B_{i,n-3}(r)] \qquad \ldots(13)$$

In the present application, an approximate solution in terms of linear combination of B-polynomials is assumed. The initial values of the unknown coefficients, a_i, are obtained with the help of Galerkin method by using the initial data. In the following sections the subscript n = N of $B_{i,N}(r)$ will be dropped for indexing simplicity.

3. Solutions

3.1 *Galerkin formulation*

equation (1) can be reduces to

$$\frac{\partial u(r,t)}{\partial t} = \frac{s}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + ze^{-ar} + g\,u$$

$$or \qquad \ldots\ldots (14)$$

$$\frac{\partial u(r,t)}{\partial t} = s\frac{\partial^2 u}{\partial r^2} + \frac{s}{r}\frac{\partial u}{\partial r} + ze^{-ar} + g\,u$$

By using the transformation $u(r,t) = h + g\,T(r,t)$,

Where

$$u(r,0) = h + g\,T(r,0) = u_0 \qquad h = \frac{W_b C_b\, T_a + Q_{m0}(1-0.1T_0)}{\rho C}$$

$$M = h + gT_w \quad \text{and} \quad g = \frac{-W_b C_b + 0.1Q_{m0}}{\rho C}$$

$$W = h + g\,T_s \qquad z = \frac{g\,S\,P\exp(a\,R_2 - 0.1)}{C}$$

with initial condition

$$u(r,0) = u_0 \,(\text{Constant}) \qquad \text{…… (15)}$$

and boundary conditions

$$u(R_1,t) = M \;\; (\text{Constant}) \qquad \text{…… (16)}$$

$$u(R_2,t) = W \;\; (\text{Constant}) \qquad \text{…… (17)}$$

We assume an approximate solution of the following form:

$$u(r,t) = f(r)\sum_{i=1}^{n} a_i(t)\,B_i(r) + \left(\frac{R_2 - r}{R_2 - R_1}\right)M + \left(\frac{r - R_1}{R_2 - R_1}\right)W \quad , \qquad \text{…… (18)}$$

which satisfies all boundary conditions

$$where\;\; f(r) = \left(\frac{R_2 - r}{R_2 - R_1}\right)\left(\frac{r - R_1}{R_2 - R_1}\right), \qquad \text{for } R_1 \le r \le R_2 \qquad \text{…… (19)}$$

$B_i(r)$ Bernstein polynomial in interval $[R_1, R_2]$ in r.

Substitution of Eq. (12) into Eq. (8) produces a residual

$$R = s\sum_i a_i(t)\ (B_i(r)\ f(r))'' + \frac{s}{r}\left(\sum_i a_i(t)\ (B_i(r)\ f(r))' + \frac{W-M}{R_2-R_1}\right) + ze^{-ar}$$
$$+ g\left(\sum_i a_i(t)\ B_i(r)\ f(r) + \left(\frac{R_2-r}{R_2-R_1}\right)M + \left(\frac{r-R_1}{R_2-R_1}\right)W\right) - \sum_i \dot{a}_i(t)\ B_i(r)\ f(r). \quad \text{...... (20)}$$

Here, $\cdot$ represents differentiation with respect to time t and ' is used to denote differentiation with respect to *x*. repeated evaluation of the inner product

$$(R, B_k) = 0, \quad k = 1, 2, ..., n \quad \text{...... (21)}$$

produces a system of ordinary differential equations that can be written as

$$\dot{A}M_5 - A(sM_1 + 2sM_2 + sM_3 + sM_4 + gM_5) - F = 0 \quad \text{...... (22)}$$

where an element of $\dot{A}$ is $\dot{a}_i$ and elements of M_1, M_2, M_3, M_4, M_5 and F are given by

$$m1_{ki} = \int_{R_1}^{R_2} B_i''(r)\, B_k(r)\ f(r)\, dr, \quad \text{...... (23)}$$

$$m2_{ki} = \int_{R_1}^{R_2} B_i''(r)\, B_k(r)\ f'(r)\, dr, \quad \text{...... (24)}$$

$$m3_{ki} = \int_{R_1}^{R_2} \frac{B_i'(r)\, B_k(r)}{r}\ f(r)\, dr, \quad \text{...... (25)}$$

$$m4_{ki} = \int_{R_1}^{R_2} B_i(r)\, B_k(r) \left(f''(r) + \frac{f'(r)}{r} \right) dr\,, \qquad \ldots\ldots (26)$$

$$m5_{ki} = \int_{R_1}^{R_2} B_i(r)\, B_k(r)\, f(r)\, dr\,, \qquad \ldots\ldots (27)$$

and

$$f_{ki} = \int_{R_1}^{R_2} B_k(r)\ dr \qquad \ldots\ldots (28)$$

The initial values of the coefficients a_i are obtained by applying the Galerkin method to the initial data

$$(u - u_0\,, B_k\,) = 0 \qquad \ldots\ldots (29)$$

i.e.

$$\left(f(r) \sum_{i=1}^{n} a_i(t)\, B_i(r) + \left(\frac{R_2 - r}{R_2 - R_1} \right) M + \left(\frac{r - R_1}{R_2 - R_1} \right) W\,, B_k \right) = (u_0, B_k)\ ,$$

Here $u_0(r) = u(r,0)$ is the initial value of $u(r,t)$.

This yields a system of equations given by

$$AM_5 = L \qquad \ldots\ldots (30)$$

Here the elements of M_5 are given by (21) and the elements of L are given by

$$l_k = \int_{R_1}^{R_2} \left(u_0 - \left(\frac{R_2 - r}{R_2 - R_1} \right) M - \left(\frac{r - R_1}{R_2 - R_1} \right) W \right) B_k \, dr \qquad \ldots\ldots (31)$$

Equation (16) can be written as

$$\dot{A} M_5 - AE - F = 0, \qquad \ldots\ldots (32)$$

Where, $E = sM_1 + 2sM_2 + sM_3 + sM_4 + gM_5$

using fourth order Runge - Kutta method for matrix is to solve it.

3.2 ***Using Homotopic Perturbation Method(HPM)*****[9,10]**

Now, replacing the domain [R1, R2] x [0, t] by a rectangular grid of points (ri, tj) with

$r_i = R_1 + il$; i = 0, 1, 2 , ……….,n

$t_j = j\Delta t$; j = 0, 1, 2, ………

We first deal with the discreatization in the space variable r and introducing

$T_i(t) = T(r_i, t)$ for t ≥ 0.

Then using central differences, equation (1) can be written in the vector – matrix form as

$$\frac{dT}{dt} = AT + B \qquad \ldots\ldots (33)$$

$$T(0) = [T_0, T_0, ..., T_0]' \qquad \text{...... (34)}$$

Where,

$$T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ . \\ . \\ . \\ T_n \end{bmatrix}; \quad A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & ... & ... & 0 \\ a_2 & b_2 & c_2 & 0 & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 0 \\ 0 & 0 & ... & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & ... & ... & 0 & a_n & b_n \end{bmatrix}; \quad B = \begin{bmatrix} a_1 T_w + f_1 \\ f_2 \\ f_3 \\ ... \\ ... \\ f_{n-1} \\ c_n T_s - f_n \end{bmatrix};$$

$$a_i = \frac{2r_i k_t - 1}{2r_i \rho c l^2}; \quad b_i = \frac{1}{\rho c}\left[\frac{Q_{m0}}{10} - W_b C_b - \frac{2k_t}{l^2}\right]; \quad c_i = \frac{2r_i k_t + 1}{2r_i \rho c l^2};$$

$$f_i = \frac{W_b C_b T_a}{\rho c} + \left(1 - \frac{T_0}{10}\right)\frac{Q_{m0}}{\rho c} + \frac{SP}{c}\exp[a(\overline{r_i} - 0.01)];$$

The homotopy of equation (33) can be constructed as

$$\frac{dT}{dt} - B = pAT, \qquad where \quad p \in [0,1] \qquad \text{...... (35)}$$

Now, Applying He's HPM technique [9] and using initial condition (34) in equation (35) we obtain the following set of differential equations in matrices:

$$\frac{dT_0}{dt} = B,$$
$$\frac{dT_1}{dt} = AT_0,$$
$$\frac{dT_2}{dt} = AT_1,$$
$$\frac{dT_3}{dt} = AT_2$$
$$\vdots$$
$$\vdots$$

and so on

Consequently, the first few components of the homotopy perturbation for equation (33) are derived as follows:

$$T_0(t) = T_0 + Bt \,, \qquad T_0 = T(0),$$
$$T_1(t) = (AT_0)t + (AB)\frac{t^2}{2},$$
$$T_2(t) = (A^2T_0)\frac{t^2}{2} + (A^2B)\frac{t^3}{6} \,,$$
$$T_3(t) = (A^3T_0)\frac{t^3}{6} + (A^2B)\frac{t^4}{24} \,,$$
$$\vdots$$
$$\vdots$$

using the above terms, we get the solution (up to four terms approximations)

$$T(t) = T_0 + Bt + (A.T_0)t + (A.B)\frac{t^2}{2} + (A^2.T_0)\frac{t^2}{2} + (A^2.B)\frac{t^3}{6} + (A^3.T_0)\frac{t^3}{6} + (A^3.B)\frac{t^4}{24}$$

4. Results and Discussion

All values of the parameters involved in the calculation are taken from Table 1.

Table 1

S	12.5 kg^{-1}
a	-127 m^{-1}
ρ	1000 kg m^{-3}
C	4180 J kg^{-1} K^{-1}
K_t	0.5 Wm^{-1}
P	20 W
W_b	8 kg m^{-3} s^{-1}
C_b	3344 J kg^{-1} K^{-1}
Q_{m0}	1.091×10^3 W m^{-3}
T_s	20°C
T_w	37°C
T_0	37°C
T_a	37°C
R_1	1×10^{-3} m
R_2	5×10^{-2} m

Comparison between numerical results of two different methods are shown in Table 2 and also depicted by graphs Fig. 2. and Fig. 3.

Table 2. (for t = 5 min.)

r	THPM	TBer
0.001	37.0	37.0
0.011	44.8	44.7396
0.021	40.2	39.9037
0.031	32.2	32.4929
0.041	26.0	25.1043
0.05	20.4	20.0

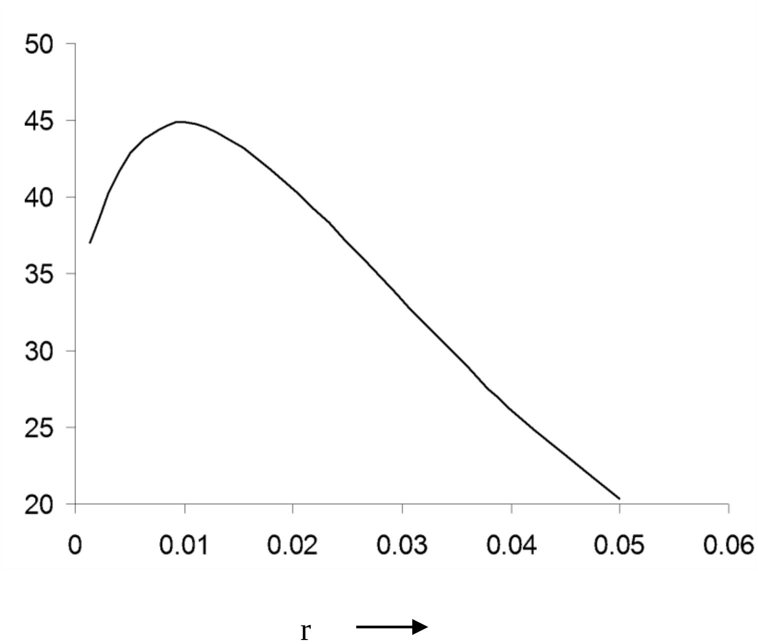

Fig.2 Plot between radius (r) Vs Temperature (T) By Homotopic perturbation method

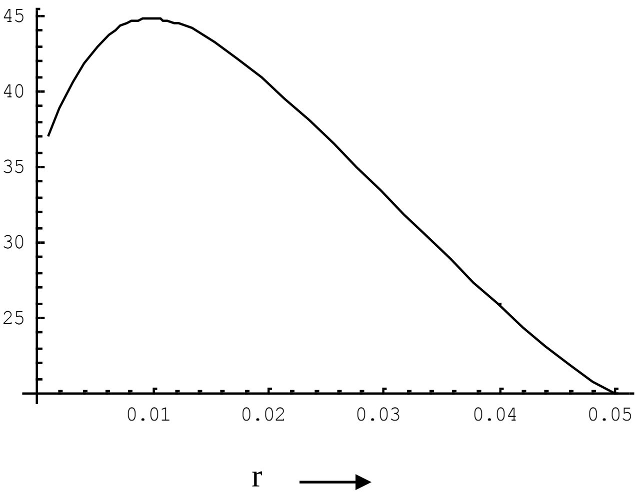

Fig.3 Plot between radius (r) Vs Temperature (T) using Bernstein polynomial Appoximation

5. Conclusion

From Figs. 2 & 3, it can be concluded that both the methods are giving identical results. During hyperthermia treatment, it is experimentally found that maximum temperature should not exceed 45°C. In our work, we find maximum temperature approx 44.8°C by both methods at the distance of 0.011m from the axis of symmetry of the tissue in time 5min.The method illustrated here gives a simple way of getting the approximate solution avoiding the appearance of ill-conditioned matrices or complicated integrations. Hence, using Bernstein polynomial for solution is in good agreement with HPM as for computational work concern for such system. This method has a potential to be used in more complex system of differential equations where exact solution is not available. Mathematica has been used for all computational works.

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Nomenclature:

a- antenna constants;

C – Specific heat of tissue

C_b - the specific heat of blood.

K_t - the thermal conductivity of the tissue

P - the transmitted power, which may be varied according to the requirements

Q_r - heat generated per unit volume of tissue due to the electromagnetic radiation absorbed

Q_b - a heat source due to blood circulation

Q_m - heat generated by the metabolic process

Q_{m0} - basal metabolic heat generation rate

r - the radial coordinate

R_1 - radius of inner tissue

R_2 - radius of outer tissue

S - the antenna constant

T – the local tissue temperature

T_s – the temperature of surface of outer tissue

T_w – the temperature of inner tissue

t – the time

W_b - the mass flow rate of the blood per unit volume of tissue

ρ- the density of tissue