APPLYING A NEW HYBRID APPROACH TO PROVIDE EXACT SOLUTIONS FOR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This paper suggests a new hybrid strategy for partial integro-differential equations arising in engineering applications. The new proposed method is based on hybridization the Kharrat-Toma integral transform with the homotopy perturbation method. This hybrid scheme aims to obtain exact solutions to several partial integro- differential equations subject to boundary or initial conditions in an effective and elegant compared to the numerical and analytical methods. In addition, that it reduces the integrals and computational steps. The obtained results display the applicability of the new suggested technique.

Keywords*: hybrid approach, Kharrat-Toma integral transform method, homotopy perturbation method, partial integro-differential equations, initial and boundary value problems, exact solution.*

INTRODUCTION

There are many physical and engineering phenomena that are modeled by initial value problems represented by partial integro-differential equations subject to initial conditions, such as reactor dynamics [1], heat conduction [2], convection-diffusion [3-6], financial pricing [7]. However, sometimes these types of problems are difficult to solve using analytical and numerical methods, therefore, the researchers suggested new hybrid techniques to solve them. Among the relative methods, we mention, Dehghan and shakeri (2010) applied variational iteration technique to solve partial integro-differential equations appearing in the conduction of heat in materials. A partial integro- differential equation appearing in financial modeling is solved by Abergel (2010). Thorwe and Bhalekar (2012) presented a Laplace transform method for most general form of linear partial integro-differential equations arising in social sciences. Hesameddini and Rahimi (2013) introduced a new numerical method includes improving of variational iteration method with Laplace transform for integro- differential equations systems. Fahim *et.al.* (2017) provided the numerical solution of a Volterra integro-differential equation via finite difference technique and sinc-collocation method.

Recently, Jafarzadeh and Keramati (2018) used a numerical method based on Taylor polynomial to obtain solutions of higher order integro differential equations. Bakodah *et.al.* (2019) suggested a method depended on the Adomian decomposition method to solve Volterra integro-differential equations and Fredholm integro-differential equations. Khan *et.al.* (2020) investigated the development of numerical methods called Cubic trigonometric B-spline for partial integro-differential equations.

This new hybrid approach aims to provide exact solutions of partial integro-differential equations that are difficult or impossible to solve by classical techniques. This hybrid approach takes advantage of the effectiveness of our new Kharrat-Toma integral transform and the semi-analytical method called homotopy perturbation method (HPM) to form a new technique with high efficiency, applicable and simple compared to other numerical and semi-analytical methods.

The remaining of the article is arranged according to the following: Section 2 provides a brief overview of the Kharrat-Toma integral transform method. Section 3 shows the methodology of the presented hybrid technique. In Section 4, the new hybrid method is tested for numerical examples. Finally, in Section 5 conclusions are outlined.

KHARRAT-TOMA INTEGRAL TRANSFORM METHOD:

The new Kharrat-Toma integral transform method is presented by Kharrat and Toma (2020) to solve initial or boundary value problems.

Definition: [16] The Kharrat-Toma integral transform technique of a function $f(x)$ is described as follows:

$$B\left[f(x)\right]=G\left(S\right)=s^{3}\int_{0}^{\infty}f(x)\,e^{\frac{-x}{s^{2}}}dx \quad , \quad x\geq 0$$

The inverse Kharrat-Toma integral transform is defined as:

$$f(x)=B^{-1}\left[G\left(S\right)\right]=B^{-1}\left[s^{3}\int_{0}^{\infty}f(x)\,e^{\frac{-x}{s^{2}}}dx\right]$$

The B^{-1} will be the inverse of the B integral transform. Where

$$B\left[f^{(n)}(x)\right]=\frac{1}{s^{2n}}G(s)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f^{(k)}(0) \quad ; \; n\geq 1$$

The fundamental properties of the Kharrat-Toma integral transform appear as follows [16]:

(1) $f(x)=1 \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=s^5$

(2) $f(x)=x^n \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=s^{2n+5}.\, n!$

THE PROPOSED HYBRID METHOD

In order to illustrate the application of the new hybrid method to solve initial and boundary value problems, let us take the following initial and boundary value problem

$$\frac{\partial^n u(x,t)}{\partial t^n}+\frac{\partial^m u(x,t)}{\partial x^m}=\psi\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},...\right)+\int_0^t K(t-s)\phi[u(x,s)]ds \qquad (1)$$

Where $u(x,t)$ unknown function, ψ and ϕ are a linear or nonlinear functions, and the kernel $K(x-t)$.

Using the Kharrat-Toma integral transform on (1), gives

$$\frac{1}{s^{2n}}B(u)-\sum_{k=0}^{n-1}s^{-2n+2k+5}\frac{\partial^k u(x,0)}{\partial t^k}=B\left[-\frac{\partial^m u(x,t)}{\partial x^m}+\psi\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},...\right)+\int_0^t K(t-s)\phi[u(x,s)]ds\right] \qquad (2)$$

Then

$$B(u)=\sum_{k=0}^{n-1}s^{2k+5}\frac{\partial^k u(x,0)}{\partial t^k}+s^{2n}B\left[-\frac{\partial^m u(x,t)}{\partial x^m}+\psi\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},...\right)+\int_0^t K(t-s)\phi[u(x,s)]ds\right] \qquad (3)$$

The homotopy of (3) can be formed as:

$$B(u)=\sum_{k=0}^{n-1}s^{2k+5}\frac{\partial^k u(x,0)}{\partial t^k}+p\,s^{2n}B\left[-\frac{\partial^m u(x,t)}{\partial x^m}+\psi\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},...\right)+\int_0^t K(t-s)\phi[u(x,s)]ds\right] \qquad (4)$$

Where $p\in[0,1]$.

According to the homotopy perturbation method the solution of (4) can be formed as follows

$$u=\sum_{i=0}^{\infty}p^i u_i \qquad (5)$$

Substituting (5) into (4), gets

$$B\left(\sum_{i=0}^{\infty}p^i u_i\right)=\sum_{k=0}^{n-1}s^{2k+5}\frac{\partial^k u(x,0)}{\partial t^k}+p\,s^{2n}B\left[\begin{array}{l}-\dfrac{\partial^m}{\partial x^m}\left(\sum_{i=0}^{\infty}p^i u_i(x,t)\right)+\psi\left(x,t,u,\dfrac{\partial}{\partial x}\left(\sum_{i=0}^{\infty}p^i u_i\right),\dfrac{\partial}{\partial t}\left(\sum_{i=0}^{\infty}p^i u_i\right),\ldots\right)\\ +\int_0^t K(t-s)\phi\left[\sum_{i=0}^{\infty}p^i u_i(x,s)\right]ds\end{array}\right] \tag{6}$$

By comparing the terms coefficients with identical powers of p in (6) and taking the inverse Kharrat-Toma integral transform, gives

$$u_i \quad ; \quad i=0,1,2,\ldots$$

Setting $p=1$, we get the approximate solution of (1)

$$u=\sum_{i=0}^{\infty}u_i(x,t)$$

TEST PROBLEMS

In this part, we present some initial and boundary value problems for partial integro- differential equations to appear the powerful and effective the new hybrid technique.

Problem .1

Let us take the linear partial linear integro- differential equation

$$u_{tt}=u_x-2e^x+2\int_0^t (t-s)\,u(x,s)ds \tag{7}$$

with the conditions

$u(x,0)=e^x \quad , \quad u_t(x,0)=0 \quad , \quad u(0,t)=\cos(t)$

Applying the Kharrat-Toma transform on (7), finds

$$\frac{1}{s^4}B(u)-s\,u(x,0)-s^3u_t(x,0)=B\left[-2e^x\right]+B\left[u_x+2\int_0^t (t-s).u(x,s)ds\right] \tag{8}$$

Then we have

$$B(u)=s^5e^x+s^4B\left[-2e^x\right]+s^4B\left[u_x+2\int_0^t (t-s).u(x,s)ds\right] \tag{9}$$

Now, constructing the homotopy on (9) as follows

$$B\left(u\right)=s^{5}e^{x}+s^{4}B\left[-2e^{x}\right]+p\,s^{4}B\left[u_{x}+2\int_{0}^{t}(t-s)\,u(x,s)ds\right] \tag{10}$$

Substituting (5) into (10), we get

$$B\left(\sum_{i=0}^{\infty}p^{i}u_{i}\right)=s^{5}e^{x}+s^{4}B\left[-2e^{x}\right]+p\,s^{4}B\left[\frac{\partial}{\partial x}\left(\sum_{i=0}^{\infty}p^{i}u_{i}\right)+2\int_{0}^{t}(t-s)\sum_{i=0}^{\infty}p^{i}u_{i}(x,s)ds\right] \tag{11}$$

Comparing coefficients of terms with identical powers of p in (11), leads to

$$p^{0}:B\left[u_{0}\right]=s^{5}e^{x}+s^{4}B\left[-2e^{x}\right] \tag{12}$$

$$p^{1}:B\left[u_{1}\right]=s^{4}B\left[\frac{\partial u_{0}}{\partial x}+2\int_{0}^{t}(t-s)\,u_{0}(x,s)ds\right] \tag{13}$$

$$p^{2}:B\left[u_{2}\right]=s^{4}B\left[\frac{\partial u_{1}}{\partial x}+2\int_{0}^{t}(t-s)\,u_{1}(x,s)ds\right] \tag{14}$$

$$\vdots$$

Taking the inverse Kharrat-Toma transform of Equations. (12), (13) and (14), yields

$$u_{0}=e^{x}-t^{2}e^{x}$$

$$u_{1}=\frac{t^{2}e^{x}}{2}-\frac{t^{6}e^{x}}{180}$$

$$u_{2}=\frac{t^{4}e^{x}}{24}+\frac{t^{6}e^{x}}{360}-\frac{t^{8}e^{x}}{10080}-\frac{t^{10}e^{x}}{453600}$$

$$u_{3}=\frac{t^{6}e^{x}}{720}+\frac{t^{8}e^{x}}{10080}-\frac{t^{12}e^{x}}{29937600}-\frac{t^{14}e^{x}}{5448643200}$$

Then the exact solution is

$$u(x,t)=u_{0}+u_{1}+u_{2}+...=\left(1-\frac{t^{2}}{2!}+\frac{t^{4}}{4!}-\frac{t^{6}}{6!}+...\right)e^{x}=\cos(t)\,e^{x}$$

Problem .2

Let us take the linear partial integro- differential equation

$$u_{xx}=u_{t}+u+2-\left(1+x^{2}\right)e^{x}+\int_{0}^{t}e^{t-s}\,u(x,s)ds \tag{15}$$

with the conditions

$u(x,0)=x^2 \quad , \quad u(0,t)=t \quad , \quad u_x(0,t)=0$

Taking the Kharrat-Toma transform on (15), finds

$$\frac{1}{s^4}B(u)-s\,u(0,t)-s^3u_t(0,t)=B\left[2-\left(1+x^2\right)e^x\right]+B\left[u_t+u+\int_0^t e^{t-s}\,u(x,s)ds\right] \tag{16}$$

Then we have

$$B(u)=s^5t+s^4B\left[2-\left(1+x^2\right)e^x\right]+s^4B\left[u_t+u+\int_0^t e^{t-s}\,u(x,s)ds\right] \tag{17}$$

Now, constructing the homotopy on (17) as follows

$$B(u)=s^5t+s^4B\left[2-\left(1+x^2\right)e^x\right]+p\,s^4B\left[u_t+u+\int_0^t e^{t-s}\,u(x,s)ds\right] \tag{18}$$

Substituting (5) into (18), we get

$$B\left(\sum_{i=0}^{\infty}p^i u_i\right)=s^5t+s^4B\left[2-\left(1+x^2\right)e^x\right]+p\,s^4B\left[\frac{\partial}{\partial t}\left(\sum_{i=0}^{\infty}p^i u_i\right)+\sum_{i=0}^{\infty}p^i u_i+\int_0^t e^{t-s}\sum_{i=0}^{\infty}p^i u_i(x,s)ds\right] \tag{19}$$

Comparing coefficients of terms with identical powers of p in (19), leads to

$$p^0: B\left[u_0\right]=s^5t+s^4B\left[2-\left(1+x^2\right)e^x\right] \tag{20}$$

$$p^1: B\left[u_1\right]=s^4B\left[\frac{\partial u_0}{\partial t}+u_0+\int_0^t e^{t-s}\,u_0(x,s)ds\right] \tag{21}$$

$$p^2: B\left[u_2\right]=s^4B\left[\frac{\partial u_1}{\partial t}+u_1+\int_0^t e^{t-s}.u_1(x,s)ds\right] \tag{22}$$

$$\vdots$$

Taking the inverse Kharrat-Toma integral transform of Equations. (20), (21) and (22), yields

$$u_0=x^2+t-\frac{x^2e^t}{2}-\frac{x^4e^t}{12}$$

$$u_1=\frac{x^2e^t}{2}-\frac{t\,x^4e^t}{24}-\frac{x^6e^t}{180}-\frac{t\,x^6e^t}{360}$$

$$u_2=\frac{t\,x^4e^t}{24}+\frac{x^4e^t}{12}-\frac{x^6e^t}{720}-\frac{t\,x^6e^t}{360}-\frac{t^2x^6e^t}{1440}-\frac{x^8e^t}{4032}-\frac{t\,x^8e^t}{5040}-\frac{t^2x^8e^t}{40320}$$

$$u_3 = \frac{t^6 e^x}{720} + \frac{t^8 e^x}{10080} - \frac{t^{12} e^x}{29937600} - \frac{t^{14} e^x}{5448643200}$$

Then the exact solution is

$$u(x,t) = x^2 + t$$

CONCLUSION

In this work, the new hybrid scheme is successfully combined the Kharrat-Toma integral transform method with the homotopy perturbation method to deal with partial integro-differential equations arising in engineering and physical applications. The new approach gives exact solutions after a few integrals and steps of calculations. Therefore, due to excellent efficiency and rapid convergence to the analytical solution, the presented technique is efficient in applying to obtain exact solutions of partial integro-differential equations.

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