DUALITIES BETWEEN FOURIER SINE AND SOME USEFUL INTEGRAL TRANSFORMATIONS

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Abstract:

The most useful technique of the mathematics which are used to finding the solutions of a lot of problems just like bending of beam, electrical network, heat related problems, which occurs in many disciplines of engineering and sciences are the techniques of integral transforms. In our research I discussed the duality between Fourier Sine transforms and some others effective integral transforms (namely Laplace transform, Mahgoub transform, Aboodh transform and Mohand transform). To justify the scope of dualities relation between Fourier Sine transform and other integral transforms (that are mentioned above, I presented the tabular representation of integral transform (namely Laplace transform, Aboodh transform, Mohand transform and Mahgoub transform) of various used functions by using Fourier Sine and other integral transforms dualities relation to signify fruitfulness of such connections. Results showed that these integral transform are strongly related with Fourier Sine transform.

Keywords: *Laplace transform, Aboodh transform, Mahgoub transform, Mohand transform, Fourier sine transform***.**

INTRODUCTION

An integral transform is a transformation [17] in the form of an integral that generates new functions based on a different variable from given functions. These transforms are of particular interest since they can be used to solve ODEs, PDEs, Fractional intedrals, Derivatives and integral equations, as well as handling and applying special functions.

An integral transform integrates a function from its original function space into a new function space, where some of the original function's properties can be better characterized and manipulated than in the original function space. Integral transforms have a wide range of uses in engineering and science, and are among the most useful techniques in mathematics [17], That are used to solve a wide range of science and engineering problems.

The fundamental goal of integral transforms is to make a difficult problem easier to solve. Laplace transform, Mahgoub transform(Laplace − Carson transform) , Aboodh transform

Mohand transform, Sumudu transform, Fourier transform, Sawi transform and Elzaki transforms are examples of integral transforms that are used to solve a variety of numerical problems.

The Fourier Transform is a useful image processing technique for breaking down an image into sine and cosine components. The image in the Fourier or frequency domain is represented by the transformation's output, while the spatial domain equivalent is represented by the input image.

Fourier sine and cosine transformations are different types of Fourier integral transform in mathematics. The Fourier sine transform (FST) and Fourier Cosine Transform (FCT) are mathematical transformations [18] that decompose functions that are dependent on space or time into functions that are dependent on spatial or temporal frequency, such as the expression of a musical chord in terms of the volumes and frequencies of its concordant, such as a musical chord's expression in terms of the volumes and frequencies of its constituent notes. The frequency domain representation and the mathematical procedure that associates the frequency domain representation with a function of space or time are referred to as the Fourier sine and cosine transform.

Odd functions are dealt with by the Fourier Sine transform. The Fourier Sine integral is used to obtain it. The Fourier Sine transform and its inverse are obtained by using the Fourier Sine integral formula [17].

The aim of this study is to establish duality relations between Fourier Sine transform with some useful integral transformation namely Laplce transforms, Aboodh transform, Mohand transform, Mahgoub (Laplace Carson) transform.

Laplace transform

The function f(x), x≥0 has a Laplace transform [20] as:

$$F(s)=L\{f(x)\}=\int_0^{\infty} e^{-st}\, f(x)\, dt. \qquad (1)$$

Aboodh transform

The function f(x), x≥0 has a Aboodh transform [6] as:

$$G(s)=A\{f(x)\}=\frac{1}{s}\int_0^{\infty} e^{-st}\, f(x)\, dt. \qquad (2)$$

Mahgoub transform

The function f(x), x≥0 has a Mahgoub transform [5] as:

$$H(s)=M_*\{f(x)\}= s\int_0^{\infty} e^{-st}\, f(x)\, dt. \qquad (3)$$

Mohand transform

The function f(x), x≥0 has a Mohand transform [21] as**:**

$$I(s)=M\{f(x)\}= s^2 \int_0^\infty e^{-st} f(x)\, dt. \tag{4}$$

FOURIER SINE TRANSFORM

Fourier Sine transformof the function f(x), x≥0 is as**:** [17]

$$F_S(K)= \mathcal{F}_s\{f(x)\})=\sqrt{\frac{2}{\pi}} \int_0^\infty sinkx f(x) dx \tag{5}$$

Dualities of Fourier Sine transform with some useful integral transforms

In this section, we define the dualities between Fourier Sine transform and some useful integral transfotm namely Laplace transform, Aboodh transform, Mahgoub transform and Mohand transform.

We first define the duality between Fourier Sine transform with above mentioned integral transforms and then we will briefly explain the duality between Fourier sine transform with these integral transforms namely Laplace transform, Aboodh transform, Mohand transform and Mahgoub transform.

A. Fourier Sine - Laplace Duality

If Fourier Sine and Laplace transform of f(x) are $F_S(K)$ and F(s)= L{f(x)} respectively then let us establish a duality relation between these transformation.

From (5),

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty sinkx f(x) dx$$

As we know $Sinkx = \frac{e^{ikx} - e^{-ikx}}{2i}$, put in (5)

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ikx} - e^{-ikx}}{2i} f(x)\, dx$$

$$F_S(K) =\sqrt{\frac{2}{\pi}} \times\frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) f(x)\, dx$$

$F_S(K) = \sqrt{\frac{1}{2\pi}} \times \frac{1}{i}\ [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$

As $\frac{1}{i} = \frac{i}{i\times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $(i^2 = -1)$

So

$F_S(K) = \sqrt{\frac{1}{2\pi}} \times -i\ [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$

$F_S(K) = \frac{i}{\sqrt{2\pi}}\ [\int_0^\infty e^{-ikx} f(x)dx - \int_0^\infty e^{ikx} f(x)dx]$

Put $ik = s\ in\ first\ term\ and \quad ik = -s_1$ in second term

$F_S(K) = \frac{i}{\sqrt{2\pi}}\ [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$

From (1)

$F_S(K) = \frac{i}{\sqrt{2\pi}}$ [L{f(x)}$-L_1${f(x)}]

Or $\frac{i}{\sqrt{2\pi}} = c$

$F_S(K)$= c [L{f(x)}$-L_1${f(x)}] (7)

or

$F_S(K)$= c [F(s) $-F_1$(s)]

Where $\frac{i}{\sqrt{2\pi}} = c$, L{f(x)}=F(s), L_1{f(x)}= F_1(s)

This is required duality of Fourier Sine transform with Laplace transform.

Example:-

- **f(x)=e^{-ax}**

Laplace transform:-

L{f(x)}=L{e^{-ax}} $= \frac{1}{S+a}$ [22]

Fourier Sine transform:-

$\mathcal{F}_s\{f(x)\} = \mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\ \left(\frac{k}{a^2+k^2}\right)$ [17]

Duality:-

From (7),

$F_S(K) = c\ [L\{f(x)\} - L_1\{f(x)\}]$

$F_S(K) = c\ [L\{e^{-ax}\} - L_1\{e^{-ax}\}]$

$= c\ \left[\frac{1}{s+a} - \frac{1}{s_1+a}\right]$

$= c\ \left[\frac{s_1+a-s-a}{(s+a)(s_1+a)}\right]$

$= c\ \left[\frac{s_1-s}{(s+a)(s_1+a)}\right]$

Put $s=ik$, $s_1 = -ik$ we get,

$= c\ \left[\frac{-ik-ik}{(ik+a)(-ik+a)}\right]$ $\quad \because (a^2-b^2) = (a-b)(a+b)$

$= c\left[\frac{-2ik}{a^2-(-ik)^2}\right]$

$= c\ \left[\frac{-2ik}{a^2-i^2k^2}\right]$

$= c\ \left[\frac{-2ik}{a^2+k^2}\right]$ $\quad \because i^2 = -1$

Here $c = \frac{i}{\sqrt{2\pi}}$, so

$= \frac{i}{\sqrt{2\pi}}\left[\frac{-2ik}{a^2+k^2}\right]$

$= \sqrt{\frac{2}{\pi}}\ \left(\frac{k}{a^2+k^2}\right)$ $\quad \because i^2 = -1$

Proved

B. Fourier Sine - Aboodh Duality

If Fourier Sine and Aboodh transform of f(x) are $F_S(K)$ and G(s)= A{f(x)} respectively then let us establish a duality relation between these transformation.

From (5)

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty sinkxf(x)dx$$

As we know $Sinkx = \frac{e^{ikx} - e^{-ikx}}{2i}$, put in (5)

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ikx} - e^{-ikx}}{2i} f(x)\, dx$$

$$F_S(K) = \sqrt{\frac{2}{\pi}} \times \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) f(x)\, dx$$

$$F_S(K) = \sqrt{\frac{1}{2\pi}} \times \frac{1}{i} [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$$

As $\frac{1}{i} = \frac{i}{i \times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $(i^2 = -1)$

So,

$$F_S(K)= \sqrt{\frac{1}{2\pi}} \times -i [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$$

$$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-ikx} f(x)dx - \int_0^\infty e^{ikx} f(x)dx]$$

Put $ik = s \; in \; first \; term \; and \quad ik = -s_1$ in second term

$$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$$

Or $\frac{i}{\sqrt{2\pi}} = c$

$$F_S(K)= c [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$$

Fs(K)= $c\ [\int_0^\infty e^{-sx} f(x)dx] - c[\int_0^\infty e^{-s_1 x} f(x)dx]$

Fs(K)= $s\ c\ [\frac{1}{s}\int_0^\infty e^{-sx} f(x)dx] - s_1\ c[\frac{1}{s_1}\int_0^\infty e^{-s_1 x} f(x)dx]$

From (2)

Fs(K)= $s\ c$ [A{f(x)}]$-s_1\ c$[A_1 {f(x)}] ……………………………….(8)

Fs(K)= c [G(s) $-G_1$(s)]

Where $\frac{i}{\sqrt{2\pi}} = c$, A{f(x)}=G(s), A_1 {f(x)}= G_1(s)

This is required duality of Fourier Sine transform with Aboodh transform.

Example:-

- **f(x)=e^{-x}cosx**

Aboodh transform:-

A{f(x)}=A{e^{-x}cosx}= $\frac{s+1}{s[(s+1)^2+1]}$

Fourier Sine transform:-

$\mathcal{F}_s$ {f(x)}= $\mathcal{F}_s\{e^{-x} cosx\}$= $\sqrt{\frac{2}{\pi}}\ \frac{k^3}{k^4+4}$

Duality:-

From (8),

Fs(K)= $s\ c$ [A{f(x)}]$-s_1\ c$[A_1 {f(x)}]

= sc [A{e^{-x}cosx}]$-s_1 c$[A_1{e^{-x}cosx}]

$$= sc\ \left[\frac{s+1}{s[(s+1)^2+1]}\right] - s_1 c\left[\frac{s_1+1}{s_1[(s_1\ +1)^2+1]}\right]$$

$$= c\ \left[\frac{s+1}{[(s+1)^2+1]}\right] - c\left[\frac{s_1+1}{[(s_1\ +1)^2+1]}\right]$$

$=c\ \left[\frac{s+1}{(s+1)^2+1}-\frac{s_1+1}{(s_1\ +1)^2+1}\right]$

Put s=ik, $s_1=-ik$ we get,

$=c\ \left[\frac{ik+1}{(ik+1)^2+1}-\frac{-ik+1}{(-ik+1)^2+1}\ \right]$

$=c\ \left[\frac{ik+1}{i^2k^2+1+2ik+1}-\frac{-ik+1}{i^2k^2+1-2ik+1}\ \right]$

$=c\ \left[\frac{ik+1}{-k^2+2ik+2}-\frac{-ik+1}{-k^2-2ik+2}\ \right]$

$=c\ \left[\ \frac{(ik+1)(-k^2-2ik+2)-(1-ik)(-k^2+2ik+2)}{(-k^2+2ik+2)(-k^2-2ik+2)}\ \right]$--*

Consider:

$$(ik+1)(-k^2-2ik+2)=(ik+1)((2-k^2)-2ik)=2ik-ik^3+2k^2+2-k^2-2ik$$

$$=-ik^3+k^2+2 \qquad \because\ i^2=-1$$

$$(ik-1)(-k^2+2ik+2)=(-ik+1)\big((2-k^2)+2ik\big)=-2ik+ik^3+2k^2+2-k^2+2ik$$

$$=ik^3+k^2+2 \qquad \because\ i^2=-1$$

$$(-k^2+2ik+2)(-k^2-2ik+2)=((2-k^2)+2ik)\big((2-k^2)-2ik\big)=(2-k^2)^2-(2ik)^2$$

$=4+k^4-4k^2+4k^2$

$=k^4+4$

Put in *

$=c\ \left[\frac{(-ik^3+k^2+2)-(ik^3+k^2+2)}{k^4+4}\right]$

$=c\ \left[\frac{-ik^3+k^2+2-ik^3-k^2-2)}{k^4+4}\right]$

Here $c=\frac{i}{\sqrt{2\pi}}$, so

$$=\frac{i}{\sqrt{2\pi}}\left[\frac{-2ik^3}{k^4+4}\right]$$

$=\sqrt{\frac{2}{\pi}}\ \frac{k^3}{k^4+4}$ ∵

$i^2 = -1$

Proved

C. Fourier Sine - Mahgoub Duality

If Fourier Sine and Mahgoub transform of f(x) are F_S(K) and H(s)= M_*{f(x)} respectively then let us establish a duality relation between these transformation.

From (5)

$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty sinkxf(x)dx$

As we know $Sinkx = \frac{e^{ikx}-e^{-ikx}}{2i}$, put in (5)

$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ikx}-e^{-ikx}}{2i} f(x)\, dx$

$F_S(K) = \sqrt{\frac{2}{\pi}} \times \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) f(x)\, dx$

$F_S(K) = \sqrt{\frac{1}{2\pi}} \times \frac{1}{i} [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$

As $\frac{1}{i} = \frac{i}{i \times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $(i^2 = -1)$

So,

$F_S(K)= \sqrt{\frac{1}{2\pi}} \times -i [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$

$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-ikx} f(x)dx - \int_0^\infty e^{ikx} f(x)dx]$

Put $ik = s\ in\ first\ term\ and$ $ik = -s_1$ in second term

$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$

Or $\frac{i}{\sqrt{2\pi}} = c$

$F_S(K)= c\ [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$

$F_S(K)= c\ [\int_0^\infty e^{-sx} f(x)dx] - c[\int_0^\infty e^{-s_1 x} f(x)dx]$

$F_S(K)= \frac{1}{s}\ c\ [\ s\int_0^\infty e^{-sx} f(x)dx] - \frac{1}{s_1} c[s_1\ \int_0^\infty e^{-s_1 x} f(x)dx]$

From (3)

$F_S(K)= \frac{1}{s}\ c\ [M_*\{f(x)\}\ ] - \frac{1}{s_1} c[M_{*1}\{f(x)\}]$ ……………………………………(9)

$F_S(K)=$ c [H(s) $-H_1$(s)]

Where $\frac{i}{\sqrt{2\pi}} = c$, $M_*\{f(x)\}$=H(s), $M_{*1}\{f(x)\}= H_1(s)$

This is required duality of Fourier Sine transform with Mahgoub transform.

Example:-

- **f(x)= xe^{-ax}**

Mahgoub transform:-

$M_*\{f(x)\}= M_*\{xe^{-ax}\}= \frac{s}{(s+a)^2}$

Fourier Sine transform:-

$\mathcal{F}_s\{f(x)\}= \mathcal{F}_s\{xe^{-ax}\}= \sqrt{\frac{2}{\pi}}\frac{2ak}{(k^2+a^2)^2}$

Duality:-

From (9),

$F_S(K)= \frac{1}{s}\ c\ [M_*\{f(x)\}\ ] - \frac{1}{s_1} c[M_{*1}\{f(x)\}]$

$F_S(K)= \frac{1}{s}\ c\ [M_*\{xe^{-ax}\}] - \frac{1}{s_1} c[M_{*1}\{xe^{-ax}\}]$

$=\frac{1}{s}c\left[\frac{s}{(s+a)^2}\right]-\frac{1}{s_1}c\left[\frac{s_1}{(s_1\ +a)^2}\right]$

$=c\left[\frac{1}{(s+a)^2}\right]-c\left[\frac{1}{(s_1\ +a)^2}\right]$

$=\mathrm{c}\ \left[\frac{1}{(s+a)^2}-\frac{1}{(s_1\ +a)^2}\right]$

$=\mathrm{c}\ \left[\frac{(s_1\ +a)^2-(s+a)^2}{(s+a)^2(s_1\ +a)^2}\right]$

Put s=ik, $s_1=-ik$ we get,

$=\mathrm{c}\ \left[\frac{(-ik+a)^2-(ik+a)^2}{(ik+a)^2(-ik+a)^2}\right]$

$=\mathrm{c}\ \left[\frac{i^2k^2+a^2-2aik-i^2k^2-a^2-2aik}{(i^2k^2+a^2+2aik)(i^2k^2+a^2-2aik)}\right]$

$=\mathrm{c}\ \left[\frac{-k^2+a^2-2aik+k^2-a^2-2aik}{(-k^2+a^2+2aik)(-k^2+a^2-2aik)}\right]$ $\because\ i^2=-1$

$=\mathrm{c}\left[\frac{-4aik}{(-k^2+\mathrm{a}^2)^2-(2\mathrm{aik})^2}\right]$ --------------------- (i) $\because\ (a^2-b^2)=(a-b)(a+b)$

Consider,

$(-k^2+\mathrm{a}^2)^2=k^4+a^4-2a^2k^2-4a^2i^2k^2$

$=k^4+a^4-2a^2k^2+4a^2k^2$ $\because\ i^2=-1$

$=k^4+a^4+2a^2k^2$

$=(k^2+\mathrm{a}^2)^2$

Put in (i)

$=\mathrm{c}\left[\frac{-4aik}{(k^2+\mathrm{a}^2)^2}\right]$

Here c= $\frac{i}{\sqrt{2\pi}}$, so

$=\frac{i}{\sqrt{2\pi}}\left[\frac{-4aik}{(k^2+\mathrm{a}^2)^2}\right]$

$=\frac{1}{\sqrt{2\pi}}\left[\frac{4aik}{(k^2+\mathrm{a}^2)^2}\right]$

$= \sqrt{\frac{2}{\pi}} \frac{2ak}{(k^2+a^2)^2}$

Proved.

D. Fourier Sine - Mohand Duality

If Fourier Sine and Mohand transform of f(x) are $F_S(K)$ and I(s)= {f(x)} respectively then let us establish a duality relation between these transformation.

From (5)

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty sinkxf(x)dx$$

As we know $Sinkx = \frac{e^{ikx}-e^{-ikx}}{2i}$, put in (5)

$$F_S(K)= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ikx}-e^{-ikx}}{2i} f(x)\, dx$$

$$F_S(K) = \sqrt{\frac{2}{\pi}} \times \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) f(x)\, dx$$

$$F_S(K) = \sqrt{\frac{1}{2\pi}} \times \frac{1}{i} [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$$

As $\frac{1}{i} = \frac{i}{i \times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $(i^2 = -1)$

So,

$$F_S(K)= \sqrt{\frac{1}{2\pi}} \times -i [\int_0^\infty e^{ikx} f(x)dx - \int_0^\infty e^{-ikx} f(x)dx]$$

$$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-ikx} f(x)dx - \int_0^\infty e^{ikx} f(x)dx]$$

Put $ik = s\ in\ first\ term\ and \quad ik = -s_1$ in second term

$$F_S(K)= \frac{i}{\sqrt{2\pi}} [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$$

Or $\frac{i}{\sqrt{2\pi}} = c$

$F_S(K) = c\ [\int_0^\infty e^{-sx} f(x)dx - \int_0^\infty e^{-s_1 x} f(x)dx]$

$F_S(K) = c\ [\int_0^\infty e^{-sx} f(x)dx] - c[\int_0^\infty e^{-s_1 x} f(x)dx]$

$F_S(K) = \frac{1}{S^2} c\ [s^2 \int_0^\infty e^{-sx} f(x)dx] - \frac{1}{{S_1}^2} c[{s_1}^2 \int_0^\infty e^{-s_1 x} f(x)dx]$

From (4)

$F_S(K) = \frac{c}{S^2}\ [M\{f(x)\}\ ] - \frac{c}{{S_1}^2} [M_1\{f(x)\}]$ ……………………………….(10)

$F_S(K) = c\ [\ I(s)\ - I_1(s)]$

Where $\frac{i}{\sqrt{2\pi}} = c$, $M\{f(x)\} = I(s)$, $M_1\{f(x)\} = I_1(s)$

This is required duality of Fourier Sine transform with Mohand transform.

Example:-

- **f(x)= sinx 0≤x≤π**

Mohand transform:-

$M\{f(x)\} = M\{\sin x\} = \frac{s^2(1+e^{-s\pi})}{(s^2+1)}$

Fourier Sine transform:-

$\mathcal{F}_s\{f(x)\} = \mathcal{F}_s\{\sin x\} = \sqrt{\frac{2}{\pi}}\frac{\sin k\pi}{1-k^2}$

Duality:-

From (10),

$F_S(K) = \frac{c}{S^2}\ [M\{f(x)\}] - \frac{c}{{S_1}^2} [M_1\{f(x)\}]$

$F_S(K) = \frac{c}{S^2}\ [M\{\sin x\}] - \frac{c}{{S_1}^2} [M_1\{\sin x\}]$

$$=\frac{c}{S^2}\left\{\frac{s^2(1+e^{-s\pi})}{(s^2+1)}\right\}-\frac{c}{{S_1}^2}\left\{\frac{{S_1}^2(1+e^{-s_1\pi})}{(s_1^2+1)}\right\}$$

$$=c\left\{\frac{1+e^{-s\pi}}{(s^2+1)}\right\}-c\left\{\frac{1+e^{-s_1\pi}}{(s_1^2+1)}\right\}$$

$$=c\left\{\frac{1+e^{-s\pi}}{s^2+1}-\frac{1+e^{-s_1\pi}}{s_1^2+1}\right\}$$

Put s=ik, $s_1=-ik$ we get,

$$=c\left\{\frac{1+e^{-ik\pi}}{(ik)^2+1}-\frac{1+e^{ik\pi}}{(-ik)^2+1}\right\}$$

$$=c\left\{\frac{1+e^{-ik\pi}}{i^2k^2+1}-\frac{1+e^{ik\pi}}{i^2k^2+1}\right\}$$

$$=c\left\{\frac{1+e^{-ik\pi}-1-e^{ik\pi}}{i^2k^2+1}\right\}$$

$$=c\left\{\frac{e^{-ik\pi}-e^{ik\pi}}{-k^2+1}\right\} \qquad \because\ i^2=-1$$

$$=-c\left\{\frac{e^{ik\pi}-e^{-ik\pi}}{-k^2+1}\right\}$$

$$=-c\left\{\frac{2isink\pi}{-k^2+1}\right\} \qquad \because\ e^{ik\pi}-e^{-ik\pi}=2isink\pi$$

Here c= $\frac{i}{\sqrt{2\pi}}$, so

$$=-\frac{i}{\sqrt{2\pi}}\left\{\frac{2isink\pi}{-k^2+1}\right\}$$

$$=\sqrt{\frac{2}{\pi}}\frac{\sin k\pi}{1-k^2} \qquad \because\ i^2=-1$$

Proved.

Applications of dualities relations (mentioned above) for finding integral transform (Fourier Sine) in tabular form.

Table 1: Fourier Sine transform of useful basic function with the help of Fourier Sine –Laplace duality relation

Sr No.	f(x)	Laplace transform	Fourier Sine transform
1	e^{-ax}	$\frac{1}{S+a}$	$\sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right)$
2	$e^{-x}\cos x$	$\frac{s+1}{(s+1)^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k^3}{k^4+4}$
3	xe^{-ax}	$\frac{1}{(s+a)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2ak}{(k^2+a^2)^2}$
4	$\sin x \quad , \quad 0\leq x\leq\pi$	$\frac{1+e^{-s\pi}}{s^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{\sin k\pi}{1-k^2}$
5	$\cos x \quad , \quad 0\leq x\leq\pi$	$\frac{s+se^{-s\pi}}{s^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k+k\cos k\pi}{k^2-1}$

6	$\begin{Bmatrix} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{Bmatrix}$	$-2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$	$\sqrt{\frac{2}{\pi}}\frac{2sink - sin2k}{k^2}$
7	$\frac{\text{sinax}}{x}$	$tan^{-1}\frac{a}{s}$	$\sqrt{\frac{1}{2\pi}}\ [log(k+a)(k-a)^{-1}]$
8	$1 \quad 0 \le x \le a$	$\frac{e^{-as}}{-S} + \frac{1}{s}$	$\sqrt{\frac{2}{\pi}}\ (\frac{1-\text{cosak}}{k})$
9	e^{-x}	$\frac{1}{S+1}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{1+k^2})$
10	xe^{-x}	$\frac{1}{(s+1)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2k}{(k^2+1)^2}$

Table 2: Fourier Sine transform of useful basic function with the help of Fourier Sine –Aboodh duality relation

Sr No.	**f(x)**	**Aboodh transform**	**Fourier Sine transform**
1	e^{-ax}	$\frac{1}{S(S+a)}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{a^2+k^2})$

2	$e^{-x}\cos x$	$\frac{s+1}{S[(s+1)^2+1]}$	$\sqrt{\frac{2}{\pi}}\frac{k^3}{k^4+4}$
3	xe^{-ax}	$\frac{1}{s(s+a)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2ak}{(k^2+a^2)^2}$
4	$\sin x \quad , \quad 0\leq x\leq\pi$	$\frac{1+e^{-s\pi}}{s(s^2+1)}$	$\sqrt{\frac{2}{\pi}}\frac{sink\pi}{1-k^2}$
5	$\cos x \quad , \quad 0\leq x\leq\pi$	$\frac{1+e^{-s\pi}}{s^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k+kcosk\pi}{k^2-1}$
6	$\begin{Bmatrix} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2 \end{Bmatrix}$	$-2\frac{e^{-s}}{s^3}+\frac{e^{-2s}}{s^3}+\frac{1}{s^3}$	$\sqrt{\frac{2}{\pi}}\frac{2sink-sin2k}{k^2}$
7	$\frac{\sin ax}{x}$	$\frac{1}{s}tan^{-1}\frac{a}{s}$	$\sqrt{\frac{1}{2\pi}}\ [log(k+a)(k-a)^{-1}]$
8	$1 \quad 0\leq x\leq a$	$\frac{e^{-as}}{-s^2}+\frac{1}{s^2}$	$\sqrt{\frac{2}{\pi}}\ (\frac{1-\cos ak}{k})$
9	e^{-x}	$\frac{1}{s(s+1)}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{1+k^2})$

10	xe^{-x}	$\frac{1}{S(s+1)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2k}{(k^2+1)^2}$

Table 3: Fourier Sine transform of useful basic function with the help of Fourier Sine –Mahgoub duality relation

Sr No.	f(x)	Mahgoub transform	Fourier Sine transform
1	e^{-ax}	$\frac{s}{S+a}$	$\sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right)$
2	$e^{-x}\cos x$	$\frac{s(s+1)}{(s+1)^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k^3}{k^4+4}$
3	xe^{-ax}	$\frac{s}{(s+a)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2ak}{(k^2+a^2)^2}$
4	$\sin x \quad , \quad 0\le x\le\pi$	$\frac{s(1+e^{-s\pi})}{(s^2+1)}$	$\sqrt{\frac{2}{\pi}}\frac{\sin k\pi}{1-k^2}$
5	$\cos x \quad , \quad 0\le x\le\pi$	$\frac{s^2(1+e^{-s\pi})}{s^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k+k\cos k\pi}{k^2-1}$

6	$\begin{Bmatrix} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & x > 2 \end{Bmatrix}$	$-2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{1}{s}$	$\sqrt{\frac{2}{\pi}} \frac{2sink - sin2k}{k^2}$
7	$\frac{\text{sinax}}{x}$	$s\, tan^{-1} \frac{a}{s}$	$\sqrt{\frac{1}{2\pi}}\ [log(k + a)(k - a)^{-1}]$
8	$1 \qquad 0 \leq x \leq a$	$1 - e^{-as}$	$\sqrt{\frac{2}{\pi}}\ (\frac{1-\cos ak}{k})$
9	e^{-x}	$\frac{s}{(S + 1)}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{1+k^2})$
10	xe^{-x}	$\frac{S}{(s + 1)^2}$	$\sqrt{\frac{2}{\pi}} \frac{2k}{(k^2 + 1)^2}$

Table 4: Fourier Sine transform of useful basic function with the help of Fourier Sine –Mohand duality relation

Sr No.	**f(x)**	**Mohand transform**	**Fourier Sine transform**
1	e^{-ax}	$\frac{s^2}{s + a}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{a^2+k^2})$

2	$e^{-x}\cos x$	$\frac{s^2(s+1)}{(s+1)^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k^3}{k^4+4}$
3	xe^{-ax}	$\frac{S^2}{(s+a)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2ak}{(k^2+a^2)^2}$
4	$\sin x \quad , \quad 0\leq x\leq\pi$	$\frac{s^2(1+e^{-s\pi})}{(s^2+1)}$	$\sqrt{\frac{2}{\pi}}\frac{\sin k\pi}{1-k^2}$
5	$\cos x \quad , \quad 0\leq x\leq\pi$	$\frac{s^3(1+e^{-s\pi})}{s^2+1}$	$\sqrt{\frac{2}{\pi}}\frac{k+k\cos k\pi}{k^2-1}$
6	$\begin{Bmatrix} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2 \end{Bmatrix}$	$-2e^{-s}+e^{-2s}+1$	$\sqrt{\frac{2}{\pi}}\frac{2\sin k-\sin 2k}{k^2}$
7	$\frac{\sin ax}{x}$	$s^2\tan^{-1}\frac{a}{s}$	$\sqrt{\frac{1}{2\pi}}\ [\log(k+a)(k-a)^{-1}]$
8	$1 \quad\quad 0\leq x\leq a$	$s(1-e^{-as})$	$\sqrt{\frac{2}{\pi}}\ (\frac{1-\cos ak}{k})$
9	e^{-x}	$\frac{s^2}{(S+1)}$	$\sqrt{\frac{2}{\pi}}\ (\frac{k}{1+k^2})$

10	xe^{-x}	$\frac{s^2}{(s+1)^2}$	$\sqrt{\frac{2}{\pi}}\frac{2k}{(k^2+1)^2}$

Conclusion

In this paper, duality relation between Fourier Sine and some useful integral transform namely Laplace transform, Aboodh transform, Mahgoub transform and Mohand transform are settled satisfyingly. Tabular representation of the integral transform (Laplace transform, Aboodh transform, Mahgoub transform(Laplace Carson transform) and Mohand transform) of many basic and useful functions are given with the succor of these dualities relation (mentioned above) to visualize the importance of these dualities between Fourier Sine and mention integral transformation. Results showed that Fourier Sine transform and mention integral transforms are strongly related to each other in this paper. In future, by using these dualities relation we can solve many advance problems related to Moderen era (such as motion of coupled harmonic oscillators, drug distribution in the body, arms race models, Brownian motion and the common health problem such as detection of diabetes.),sciences and engineering.

References:

[1] Deakin, M. A. (1985). Euler's invention of integral transforms. *Archive for history of exact sciences*, *33*(4), 307-319.

[2] Deakin, M. A. (1981). The development of the Laplace transform, 1737–1937. *Archive for History of Exact sciences*, *25*(4), 343-390.

[3] G. K. Watugala, "Sumudu transform: a new integral transform to solve differential equations and control engineering problems," International Journal of Mathematical Education inScience and Technology, volume 24 (1993), 35–43.

[4] Thao, N. X., Kakichev, V. A., & Tuan, V. K. (1998). On the generalized convolutions for Fourier cosine and sine transforms. *East-West J. Math*, *1*(1), 85-90.

[5] Taha, N. E. H., Nuruddeen, R. I., & Sedeeg, A. K. H. (2017). Dualities between"Kamal & Mahgoub Integral Transforms"and"Some Famous Integral Transforms". *Current Journal of Applied Science and Technology*, 1-8.

[6] Aboodh, K. S. (2013). The New Integral Transform'Aboodh Transform. *Global Journal of Pure and Applied Mathematics*, *9*(1), 35-43.

[7] Elzaki, T. M. (2011). The new integral transform 'Elzaki transform'. *Global Journal of pure and applied mathematics*, *7*(1), 57-64.

[8] Chaudhary, P., Chanchal, P., Khandelwal, Y., & Singh, Y. (2018). Duality of Some Famous Integral Transforms from the polynomial Integral Transform. *International Journal of Mathematics Trends and Technology*, *55*(5), 345-349.

[9] Ike, C. C. (2018). Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation. *International Journal of Darshan Institute on Engineering Research and Emerging Technologies (IJDI-ERET), 7 (1)*, 1-6.

[10] Aggarwal, S., & Gupta, A. R. (2019). Dualities between Mohand transform and some useful integral transforms. *International Journal of Recent Technology and Engineering*, *8*(3), 843-847.

[11] Chauhan, R., Kumar, N., & Aggarwal, S. (2019). Dualities between Laplace-Carson transform and some useful integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, *8*(12), 1654-1659.

[12] Aggarwal, S., & Gupta, A. R. (2019). Dualities between some useful integral transforms and Sawi transform. *International Journal of Recent Technology and Engineering*, *8*(3), 5978-5982.

[13] Aggarwal, S., & Bhatnagar, K. (2019). Dualities between Laplace transform and some useful integral transforms. *International Journal of Engineering and Advanced Technology*, *9*(1), 936-941.

[14] Aggarwal, S., Bhatnagar, K., & Dua, A. (2019). Dualities between Elzaki transform and some useful integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, *8*(12), 4312-4318.

[15] Chaudhary, R., Sharma, S. D., Kumar, N., & Aggarwal, S. Connections between Aboodh Transform and Some Effective Integral Transforms.

[16] L. Debnath, D. Bhatta, Integral Transforms and their application, 3rd Edition page(1-104) (2015)

[17] Kreyszig, E. (2009). Advanced Engineering Mathematics 10th Edition.

[18] Debnath, L., & Bhatta, D. (2014). *Integral transforms and their applications*. CRC press.

[19] Dr. Nawazish, Partial Differential Equation, 1st Edition, page (681-688), (2013) http://www.aonepublishers.com/products/Partial-Differential-Equations.html

[20] Bellman, R., & Roth, R. S. (1984). *The laplace transform* (Vol. 3). World Scientific.

[21] Aggarwal, S., & Chauhan, R. (2019). A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*, *7*(1), 520-529.

[22] S. M. Yousaf, A. Majeed, M. Amin Mathematical Methods, chapter 11, Laplace Transform.