# SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION WITH DIRICHLET CONDITIONS USING TECHNIQUES OF THE INVERSE MOMENTS PROBLEM

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## Abstract

It will be shown that finding solutions from some integro-differential equation under Dirichlet conditions is equivalent to solving an integral equation, which can be treated as a generalized two-dimensional moment problem over a domain  $E = \{(x, t), 0 < x < L; t > 0\}$ . We will see that an approximate solution of the equation integro-differential can be found using the techniques of generalized inverse moments problem and bounds for the error of the estimated solution. First the problem is reduced to solving a hyperbolic or parabolic partial derivative equation considering the unknown source. The method consists of two steps. In each one an integral equation is solved numerically using the two-dimensional inverse moment problem techniques.

We illustrate the different cases with examples.

Keywords: integro-differential equation, integral equations, generalized moment problem.

## **INTRODUCTION**

Integral and integro-differential equations are found in numerous applications in different fields of science and engineering. For instance, in the mathematical modelling of spatio temporal developments, epidemic modelling and various biological and physical problems. Analytical solutions of integral and integro-differential equations, however, either do not exist or it is often hard to find. It is precisely due to this fact that several numerical methods have been developed for finding approximate solutions of integral and integro-differential equations.

The issue of solving different types of integro-differential equations has been widely discussed in the literature and a great variety of methods have been proposed for its numerical resolution.

Biorthogonal spline wavelet method is proposed for the numerical solution of Linear and nonlinear integral and integro-differential equations in [1]. In [2] it is proposed a new hybrid method to find

analytical approximate or exact solutions for various many linear and nonlinear integro – differential equations. In [3] the paper presents an iterative technique based on homotopy analysis method for solving system of Volterra integro-differential equations. The technique provides us series solutions to the problems which are combined with the diagonal Padé approximants and Laplace transform to obtain closed-form solutions. In [4] the paper is concerned with modification of the Adomian Decomposition Method for solving linear and non-linear Volterra and Volterra-Fredholm Integro-Differential equations. In [5] this paper, we introduced the modified differential transform which is a modified version of a two-dimensional differential transform method. In [6] different classes of integral and integro-differential equations are solved using a modified differential transform method. This proposed technique is based on differential transform method (DTM), Laplace transform (LT) procedure and Padé approximants (PA). In [7] a new modification of homotopy perturbation method was proposed to find analytical solution of high-order integrodifferential equations. The Modification process yields the Taylor series of the exact solution. Canonical polynomials are used as basis function equations. The Modification process yields the Taylor series of the exact solution. Canonical polynomials are used as basis function. [8] presents an effective hybrid semi-analytical method for dealing with the integro-differential equations. This new technique is based on the combining of the Kharrat-Toma integral transform with the homotopy perturbation method to find the exact or approximate solutions of both linear and nonlinear models. In [9] a combination between a Sumudu transform (ST) and the homotopy perturbation method (HPM) is presented. Other recent works are [10, 11], to name a few.

Some study inverse boundary problems for one dimensional linear integro-differential equation of the Gurtin – Pipkin type with the Dirichlet to Neumann map as the inverse data [12].

Other jobs study the stability of solutions for a heat equation with memory [13].

In [14] study decay properties in energy norm for solutions of a class of partial differential equations with memory are studied by means of frequency domain methods.

In [15] it is tested that the one-dimensional heat equation with memory cannot be controlled to rest for large classes of memory kernels and controls. The approach is based on the application of the theory of interpolation in Paley–Wiener spaces.

In this work we want to find w(x, t) such that

$$w_t(x,t) = \int_0^t k(t-s) w_{xx}(x,s) ds + f(x,t) \, .$$

about a domain  $E = \{(x, t), 0 < x < L; t > 0\}$  with conditions

$$w(x, 0) = h_1(x);$$
  $w(0, t) = k_1(t);$  ;  $w(L, t) = k_2(t).$ 

where k(t) has continuous derivate on x = 0, the value of k(0) is known, f(x, t) known and derivative with respect to t continuous, using the problem generalized moments techniques.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. It is not the objetive to compare with other methods.

The generalized moments problem [16, 17, 18], is to find a function f(x) about a domain  $\Omega \subset \mathbb{R}^d$  that satisfies the sequence of equations

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i \in \mathbb{N} - - - - - - - (1)$$

Where N is the set of the natural numbers,  $(g_i(x))$  is a given sequence of functions in  $L^2(\Omega)$  linearly independent known and the succession of real numbers  $\{\mu_i\}_{i \in N}$  are known data. The problem of Hausdorff moments [17,18], is to find a function f(x) in (a, b) such that

$$\mu_i = \int_a^b x^i f(x) dx \quad i \in \mathbb{N} \,.$$

In this case  $g_i(x) = x^i$  with *i* belonging to set *N*.

If the integration interval is  $(0, \infty)$  we have the problem of Stieltjes moments; if the integration interval is  $(-\infty, \infty)$  we have the problem of Hamburger moments [16,17].

The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [16,17,18]. There are several methods to build regularized solutions. One of them is the truncated expansion method [17].

This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i = 1, 2, \dots, n. - - - - - (2)$$

where it is considered as approximate solution of f(x) to  $p_n(x) = \sum_{i=0}^n \lambda_i \phi_i(x)$ , and the functions  $\{\phi_i(x)\}_{i=1,\dots,n}$  result of orthonormalize  $\langle g_1, g_2, \dots, g_n \rangle$  being  $\lambda_i$  the coefficients based on the data  $\mu_i$ . In the subspace generated by  $\langle g_1, g_2, \dots, g_n \rangle$  the solution is stable. If  $n \in N$  is chosen in an appropriate way then the solution of (2) it approaches the solution of the problem (1).

In the case where the data  $\mu_i$  are inaccurate the convergence theorems should be applied and error estimates for the regularized solution (pages 19 - 30 of [17]).

# **ARTICLE ORGANIZATION**

To find w(x, t) such that

$$w_t(x,t) = \int_0^t k(t-s) w_{xx}(x,s) ds + f(x,t) \, .$$

about a domain  $E = \{(x, t), 0 < x < L; t > 0\}$ with conditions

$$w(x, 0) = h_1(x);$$
  $w(0, t) = k_1(t);$  ;  $w(L, t) = k_2(t).$ 

we will do it in two steps.

The next section describes the first step.

Then it is explained how the generalized moment problem is solved with the truncated expansion method.

The section that follows explains the second step.

Finally the numerical example and the conclusions.

## **RESOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION – FIRST STEP**

We want to find w(x, t) such that

about a domain  $E = \{(x, t), 0 < x < L; t > 0\}.$ 

We derive with respect to *t*:

$$w_{tt}(x,t) = \int_0^t k_t(t-s) w_{xx}(x,s) ds + k(t-t) w_{xx}(x,t) + f_t(x,t).$$

Then

$$w_{tt}(x,t) - k(0)w_{xx}(x,t) = \int_0^t k_t(t-s)w_{xx}(x,s)ds + f_t(x,t).$$

We considerer

$$w_{tt}(x,t) - k(0)w_{xx}(x,t) = G(x,t) - - - - - (4)$$

We can solve (4) as a Klein-Gordon equation with Dirichlet conditions where G(x, t) is unknown.

We consider as auxiliary function

$$u(m,r,x,t) = e^{-m(x+1)}e^{-r(t+1)}$$
.

We write k = k(0)

and

$$w(x,0) = h_1(x);$$
  $w(0,t) = k_1(t);$  ;  $w(L,t) = k_2(t)$ .

We define the vector field

$$F^* = (F_1(w), F_2(w)) = (-kw_x, w).$$

As  $u \operatorname{div}(F^*) = u G(x, t)$  we have to:

$$\iint_E u \, div(F^*) dA = \iint_E u \, G(x,t) dA.$$

Moreover, as  $u \operatorname{div}(F^*) = \operatorname{div}(u F^*) - F^* \cdot \nabla u$ , then

$$\iint_E u \, div(F^*) dA = \iint_E div(u \, F^*) dA - \iint_E F^* \, \nabla u \, dA - - - - - 5$$

where  $\nabla u = (u_x, u_t)$ .

Besides that

$$\iint_{E} div(u F^{*}) dA = \iint_{E} (-k u w_{x})_{x} + (u w_{t})_{t} dA = \_\_\__6$$
$$\iint_{E} u div(F^{*}) dA + \iint_{E} (-k u_{x} w_{x} + u_{t} w_{t}) dA.$$

Then from (5) and (6):

$$\iint_E (-k u_x w_x + u_t w_t) dA = \iint_E F^* \cdot \nabla u \, dA \cdot - - - - - - - 7$$

On the other hand, it can be proven, after several calculations that, integrating by parts:

$$\iint_{E} F^{*} \cdot \nabla u \, dA = A(m,r) + B(m,r) - \iint_{E} u \, w \, (-k \, m^{2} + r^{2}) dA = \varphi(m,r) - - - -8$$

with

$$A(m,r) = \int_0^\infty (-m)(-k)u(m,r,L,t)w(L,t) - (-m)(-k)u(m,r,0,t)w(0,t)dt$$
$$B(m,r) = \int_0^L - (-r)u(m,r,x,0)w(x,0)dx.$$

If  $\sqrt{k} m = r$ , instead of (7) and (8)

$$\iint_{E} (-k)(-m)u w_{x} - \sqrt{k} m w_{t} u dA = \varphi(m, \sqrt{k} m)$$
  
$$\therefore \quad \iint_{E} u (kw_{x} - \sqrt{k} w_{t}) dA = \frac{\varphi(m, \sqrt{k} m)}{m}.$$

with

$$\frac{\varphi(m,\sqrt{k}\,m)}{m} = \int_0^\infty k\,u(m,\sqrt{k}\,m,L,t)w(L,t) - k\,u(m,\sqrt{k}\,m,0,t)w(0,t)dt + \int_0^L u(m,\sqrt{k}\,m,x,0)\sqrt{k}\,w(x,0)dx \;.$$

We note  $\varphi_1(m) = \frac{\varphi(m,\sqrt{k}m)}{m}$ , then

$$\iint_E u(kw_x - \sqrt{k} \ w_t) dA = \varphi_1(m) \dots - - - - - - - - 9$$

To solve this integral equation we take a base  $\psi_i(m) = m^i e^{-m}$  i = 0, 1, 2, ..., nThen we multiply both members of (9) by  $\psi_i(m) = m^i e^{-m}$  and we integrate with respect to *m*, we obtain

$$\iint_{E} H_{i}(x,t) (kw_{x} - \sqrt{k} w_{t}) dA = \int_{0}^{\infty} \varphi_{1}(m) \psi_{i}(m) dm = \mu_{i} \quad i = 0, 1, 2, \dots, n \dots - - - 10$$

where  $H_i(x,t) = \int_0^\infty u(m,\sqrt{k} m, x, t)\psi_i(m)dm$ .

We can interpret (10) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation  $p_n(x, t)$  for  $kw_x - \sqrt{k} w_t$ .

#### SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [18] and generalized in [15] and [19] to find an approximation  $p_n(x,t)$  of  $kw_x - \sqrt{k} w_t$  for the corresponding finite problem with i = 0,1,2,...,n where *n* is the number of moments  $\mu_i$ . We consider the basis  $\phi_i(x,t)$  i = 0,1,2,...,n obtained by applying the Gram-Schmidt orthonormalization process on  $H_i(x,t)$  i = 0,1,2,...,n.

We approximate the solution  $kw_x - \sqrt{k} w_t$  with [18] and generalized in [19] and [20]:

$$p_n(x,t) = \sum_{i=0}^n \lambda_i \phi_i(x,t) \quad \text{where} \quad \lambda_i = \sum_{j=0}^i C_{ij} \ \mu_j \quad i = 0,1,2,\dots,n \,.$$

And the coefficients  $C_{ij}$  verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) | \phi_k(x,t) \rangle}{\| \phi_k(x,t) \|^2} C_{kj}\right) \cdot \| \phi_i(x,t) \|^{-1} \quad 1 < i \le n \; ; 1 \le j < i \; .$$

The terms of the diagonal are  $\|\phi_i(x, t)\|^{-1}$  i = 0, 1, ..., n.

The proof of the following theorem is in [20,21]. In [21] the demonstration is made for  $b_2$  finite. If  $b_2 = \infty$  instead of taking the Legendre polynomials we take the Laguerre polynomials. En [22] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

#### Theorem

Sea  $\{\mu_i\}_{i=0}^n$  be a set of real numbers and suppose that  $f(x,t) \in L^2((a_1,b_1) \times (a_2,b_2))$  for two positive numbers  $\varepsilon$  and M verify:

$$\sum_{i=0}^{n} \left| \iint_{E} H_{i}(x,t)f(x,t)dtdx - \mu_{i} \right|^{2} \le \varepsilon^{2} \quad .$$
$$\iint_{E} (x f_{x}^{2} + t f_{t}^{2}) Exp[x+t]dtdx \le M^{2} \quad .$$

then

$$\int_{a_1}^{b_1} \int_{a_2}^{\infty} |f(x,t)|^2 dt dx \le \min_i \left\{ \|C^T C\| \varepsilon^2 + \frac{1}{8(n+1)^2} M^2; i = 0, 1, \dots, n \right\}$$

and

$$\int_{a_1}^{b_1} \int_{a_2}^{\infty} |f(x,t) - p_n(x,t)|^2 \, dt dx \le \|C^T C\|\varepsilon^2 + \frac{1}{8(n+1)^2} M^2.$$

And it must be fulfilled that

$$t^i f(x,t) \to 0$$
 si  $t \to \infty$  para todo  $i \in N$ .

#### **RESOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION – SECOND STEP**

So we have an equation in first order partial derivatives of the form

$$kw_x(x,t) - \sqrt{k}w_t(x,t) = p_n(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t) + A_2(x,t)w_t(x,t) = p_n(x,t).$$

where  $A_1(x,t) = k$  and  $A_2(x,t) = -\sqrt{k}$ 

It is resolved as in [21], that is, we can prove that solving this equation is equivalent to solving the integral equation

with  $K(m, r, x, t) = u(m, r, x, t) \left( -m_1 k(m+1) + m_2 \sqrt{k}(r+1) \right)$ 

where now it is taken as an auxiliary function

$$u(m,r,x,t) = e^{-m_1(m+1)(x+1)}e^{-m_2(r+1)(t+1)}$$

The values of  $m_1$  and  $m_2$  are chosen in a convenient way to avoid discontinuities, and

$$\varphi_2(m,r) = \int_0^L u(m,r,x,0)\sqrt{k} w(x,0)dx + \int_0^\infty u(m,r,L,t)k w(L,t) - u(m,r,0,t)k w(0,t)dt - \int_0^\infty \int_0^L p_n(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)}$$
  $i = 0,1,2,...,n_1$   $j = 0,1,2,...,n_2$ .

And we multiply both members of (11) by  $\psi_{ij}(m, r)$  and we integrate with respect to *m* and *r*.

We have then the generalized moments problem

$$\int_0^L \int_0^\infty w(x,t) H_i(x,t) \ dt dx = \mu_{ij}$$

where

$$\mu_{\{ij\}} = \int_{\{0\}}^{\{L\}} \int_0^\infty \varphi_2(m, r) \psi_{ij}(m, r) dm dr .$$
$$H_{ij}(x, t) = \int_0^L \int_0^\infty K(m, r, x, t) \psi_{ij}(m, r) dm dr$$

We apply the truncated expansion method and find a numerical approximation for w(x, t). We can solve in an analogous way the equations

$$w_{tt} = aw_{xx} + \int_0^t k(t-s)w_{xx}(x,s)ds + f(x,t) - - - - - - 12$$

and

$$w_t = w_{xx} + \int_0^t k(t-s)w_{xx}(x,s)ds + f(x,t) - - - - - - - 13$$

In both cases the domain is  $E = \{(x, t), 0 < x < L; t > 0\}$  with Dirichlet conditions.

If (12) the equation would be solved

$$w_{tt} - aw_{xx} = G(x,t),$$

if (13) the equation would be solved

$$w_{tt} - w_{xx} = G(x, t) \; .$$

with 
$$G(x,t) = \int_0^t k(t-s)w_{xx}(x,s)ds + f(x,t)$$
 where  $G(x,t)$  unknown.

This last case was resolved in [23].

#### NUMERICAL EXAMPLE:

We consider the equation

$$w_t = \int_0^t \cos(t-s) w_{xx}(x,s) ds - \frac{1}{5} e^{-1-x-2t} (8 + 2e^{2t} \cos(t) + e^{2t} \sin(t))$$

in  $(0,3) \times (0,\infty)$ .

Conditions:

$$w(0,t) = e^{-2t-1}$$
  $w(3,t) = e^{-4-2t}$   $w(x,0) = e^{-x-1}$ .

The solution is:  $w(x, t) = e^{-2t-1-x}$ 

For the first step we take n = 5 moments and we approximate  $w_x(x, t) - w_t(x, t) = G1(x, t)$ .

with accuracy 
$$\int_0^3 \int_0^\infty (p_{15}(x,t) - G1(x,t))^2 = 0.0324621.$$

In the Fig. 1 we show  $p_{15}(x, t)$  and G1(x, t) overlapping.



Fig. 1  $p_{15}(x, t)$  and G1(x, t)

For the second step we take  $m_1 = m_2 = 1$ . We also consider  $n_1 = 3$  and  $n_2 = 2$ , that is 6 moments.

We approximate w(x,t) with accuracy  $\int_{0}^{3} \int_{0}^{\infty} (p_{26}(x,t) - w(x,t))^{2} = 0.0135942.$ 

In the Fig. 2 we show  $p_{26}(x, t)$  and w(x, t) overlapping.



Fig. 2  $p_{26}(x, t)$  and w(x, t)

## CONCLUSION

An equation integro-differential of the form  $w_t(x,t) = \int_0^t k(t-s) w_{xx}(x,s) ds + f(x,t)$  where the unknown function w(x,t) is defined in  $(0,L) \times (0,\infty)$  under the Dirichlet conditions can be solved numerically by applying inverse problem techniques of moments in two steps considering the equation in partial derivatives  $w_{tt}(x,t) - k(0)w_{xx}(x,t) = G(x,t)$  where k(0) is known and G(x,t) unknown.

1. First we consider the integral equation

$$\iint_E u \big( k w_x - \sqrt{k} \, w_t \big) dA = \varphi_1(m) \, .$$

we can solve it numerically as a inverse moments problem, and we get an approximate solution for  $kw_x(x,t) - \sqrt{k} w_t(x,t)$ .

2. as a second step we consider the integral equation

$$\int_0^L \int_0^\infty K(m,r,x,t) w(x,t) dt dx = \varphi_2(m,r) \, .$$

and again it can be solved numerically by applying inverse moments problem techniques, and we get an approximate solution for w(x, t).

We can solve in an analogous way the equations  $w_{tt} = aw_{xx} + \int_0^t k(t-s)w_{xx}(x,s)ds + f(x,t)$ and  $w_t = w_{xx} + \int_0^t k(t-s)w_{xx}(x,s)ds + f(x,t)$ .

In both cases the domain is  $(0, L) \times (0, \infty)$  under the Dirichlet conditions.

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