

APPLICATION OF ZZ-TRANSFORM WITH HOMOTOPY PERTURBATION METHOD FOR SOLVING HIGHER-ORDER BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, a new hybrid scheme for solving the higher-order boundary value problems is introduced. the presented approach is based on combination of the ZZ-transform method with the homotopy perturbation method to find the solutions of boundary value problems. To show the power of the proposed technique, several examples are tested and the comparison results are tabulated.

Keywords: *ZZ-transform method, homotopy perturbation method, hybrid scheme, higher-order, boundary value problems.*

INTRODUCTION

Higher-order boundary value problems arise in many engineering and applied sciences applications, therefore the researchers have been interested in this type of problems. there are many numerical and analytical methods used to solve the higher-order boundary value problems such as Adomian decomposition method [1,2], differential transform technique [3], Quintic B-spline collocation method [4], chebychev polynomial technique [5], iterative method [6], developed Euler-collocation method [7] and variational iteration method [8].

The aim of this paper is to suggest a new hybrid method combines the ZZ-transform with the homotopy perturbation method to find the solutions of higher-order boundary value problems that are difficult to solve via classical methods. the new proposed technique exploits the advantage of the ZZ-transform and the homotopy perturbation method (HPM) to introduce a new scheme with high accuracy.

The organization of the paper is as follows. In Section 2, the definition of ZZ-transform is introduced. In Section 3, the proposed hybrid method is presented. In section 4, some numerical examples are tested. Finally, Section 5 provides conclusions of the work.

DEFINITION OF ZZ-TRANSFORM

In 2016, the integral transform "ZZ-transform method" was introduced by Zain Abadin Zafar [9]. ZZ-transform technique is similar to the other transforms such as Sumudu and Laplace transforms.

Definition [9]:

let $f(x)$ be a function defined for all $x \geq 0$. The ZZ-transform of a function $f(x)$ is described as follows

$$H(f(x)) = Z(v, s) = s \int_0^{\infty} f(vx)e^{-sx} dx$$

THE PROPOSED SCHEME

In this section, the methodology of the presented method is explained by detail as follows:

Consider the following n-th order boundary value problem

$$v^{(n)}(x) = f(x, v, v', \dots, v^{(n-1)}) \quad (1)$$

With boundary conditions

$$v(0) = \alpha_0, v'(0) = \alpha_1, \dots, v^{(n-1)}(0) = \alpha_{n-1}$$

Where f is a continuous function on $[0,1] \times D$, D is an open subset of \mathbb{R}^{n-1} and α_i are real numbers.

Taking the ZZ- transform on (1), we find

$$\frac{s^n}{v^n} H(v) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{v^{n-k}} v^{(k)}(0) = H\left[f(x, v, v', \dots, v^{(n-1)})\right]$$

Then we have

$$H(v) = \frac{v^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k}}{v^{n-k}} v^{(k)}(0) + \frac{v^n}{s^n} H\left[f(x, v, v', \dots, v^{(n-1)})\right] \quad (2)$$

Now, we construct the homotopy on (2) yields

$$H(v) = \frac{v^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k}}{v^{n-k}} v^{(k)}(0) + p \frac{v^n}{s^n} H\left[f(x, v, v', \dots, v^{(n-1)})\right] \quad (3)$$

Where $p \in [0,1]$ is an embedding parameter.

According to the homotopy perturbation method the solution of (3) can be written as a power series in p

$$v = \sum_{i=0}^{\infty} p^i v_i \quad (4)$$

Substituting (4) into (3), we get

$$H \left(\sum_{i=0}^{\infty} p^i v_i \right) = \frac{v^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k}}{v^{n-k}} v^{(k)}(0) + p \frac{v^n}{s^n} H \left[f \left(x, \sum_{i=0}^{\infty} p^i v_i, \sum_{i=0}^{\infty} p^i v'_i, \dots, \sum_{i=0}^{\infty} p^i v_i^{(n-1)} \right) \right]$$

Comparing the coefficients of terms with identical powers of p and taking the inverse ZZ-transform and we get the approximate solution of (1) where $p = 1$

$$v = \sum_{i=0}^{\infty} v_i$$

NUMERICAL EXAMPLES

In this section, to test the accuracy of the proposed method several examples are presented

Example .1

Consider the following ninth-order boundary value problem:

$$v^{(9)}(x) = -9e^x + v(x) \quad , \quad 0 \leq x \leq 1 \quad (5)$$

With boundary conditions

$$v^{(i)}(0) = 1 - i \quad ; \quad i = 0, 1, 2, 3, 4$$

$$v^{(i)}(1) = -i e \quad ; \quad i = 0, 1, 2, 3$$

Taking the ZZ- transform on (5), finds

$$\begin{aligned} & \frac{s^9}{v^9} H(v) - \frac{s^9}{v^9} v(0) - \frac{s^8}{v^8} v'(0) - \frac{s^7}{v^7} v''(0) - \frac{s^6}{v^6} v^{(3)}(0) - \frac{s^5}{v^5} v^{(4)}(0) - \frac{s^4}{v^4} v^{(5)}(0) - \frac{s^3}{v^3} v^{(6)}(0) \\ \text{Then } & -\frac{s^2}{v^2} v^{(7)}(0) - \frac{s}{v} v^{(8)}(0) = -9H \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \right) + H(v) \end{aligned}$$

we have

$$\begin{aligned} H(v) &= 1 - \frac{v^2}{s^2} - 2 \frac{v^3}{s^3} - 3 \frac{v^4}{s^4} + \alpha_1 \frac{v^5}{s^5} + \alpha_2 \frac{v^6}{s^6} + \alpha_3 \frac{v^7}{s^7} + \alpha_4 \frac{v^8}{s^8} \\ & - 9 \left(\frac{v^9}{s^9} + \frac{v^{10}}{s^{10}} + \frac{v^{11}}{s^{11}} + \frac{v^{12}}{s^{12}} + \frac{v^{13}}{s^{13}} + \frac{v^{14}}{s^{14}} + \frac{v^{15}}{s^{15}} \right) + \frac{v^9}{s^9} H(v) \end{aligned} \quad (6)$$

Now, we construct the homotopy on (6) then we find

$$H(v) = 1 - \frac{v^2}{s^2} - 2\frac{v^3}{s^3} - 3\frac{v^4}{s^4} + \alpha_1\frac{v^5}{s^5} + \alpha_2\frac{v^6}{s^6} + \alpha_3\frac{v^7}{s^7} + \alpha_4\frac{v^8}{s^8} \\ - 9\left(\frac{v^9}{s^9} + \frac{v^{10}}{s^{10}} + \frac{v^{11}}{s^{11}} + \frac{v^{12}}{s^{12}} + \frac{v^{13}}{s^{13}} + \frac{v^{14}}{s^{14}} + \frac{v^{15}}{s^{15}}\right) + p\frac{v^9}{s^9}H(v) \quad (7)$$

Substituting (4) into (7), we get

$$H\left(\sum_{i=0}^{\infty} p^i v_i\right) = 1 - \frac{v^2}{s^2} - 2\frac{v^3}{s^3} - 3\frac{v^4}{s^4} + \alpha_1\frac{v^5}{s^5} + \alpha_2\frac{v^6}{s^6} + \alpha_3\frac{v^7}{s^7} + \alpha_4\frac{v^8}{s^8} \\ - 9\left(\frac{v^9}{s^9} + \frac{v^{10}}{s^{10}} + \frac{v^{11}}{s^{11}} + \frac{v^{12}}{s^{12}} + \frac{v^{13}}{s^{13}} + \frac{v^{14}}{s^{14}} + \frac{v^{15}}{s^{15}}\right) + p\frac{v^9}{s^9}H\left(\sum_{i=0}^{\infty} p^i v_i\right) \quad (8)$$

Comparing coefficients of terms with identical powers of p in (8), we get

$$p^0 : H(v_0) = 1 - \frac{v^2}{s^2} - 2\frac{v^3}{s^3} - 3\frac{v^4}{s^4} + \alpha_1\frac{v^5}{s^5} + \alpha_2\frac{v^6}{s^6} + \alpha_3\frac{v^7}{s^7} + \alpha_4\frac{v^8}{s^8} \\ - 9\left(\frac{v^9}{s^9} + \frac{v^{10}}{s^{10}} + \frac{v^{11}}{s^{11}} + \frac{v^{12}}{s^{12}} + \frac{v^{13}}{s^{13}} + \frac{v^{14}}{s^{14}} + \frac{v^{15}}{s^{15}}\right) \quad (9)$$

$$p^1 : H(v_1) = \frac{v^9}{s^9} H(v_0) \quad (10)$$

Taking the inverse ZZ- transform of (9) and (10) yields

$$v_0 = 1 - \frac{x^2}{2!} - 2\frac{x^3}{3!} - 3\frac{x^4}{4!} + \alpha_1\frac{x^5}{5!} + \alpha_2\frac{x^6}{6!} + \alpha_3\frac{x^7}{7!} + \alpha_4\frac{x^8}{8!} \\ - 9\left(\frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \frac{x^{13}}{13!} + \frac{x^{14}}{14!} + \frac{x^{15}}{15!}\right)$$

$$v_1 = \frac{x^9}{362880} - \frac{x^{11}}{39916800} - \frac{x^{12}}{239500800} - \frac{x^{13}}{2075673600} + \alpha_1 \frac{x^{14}}{87178291200} \\ + \alpha_2 \frac{x^{15}}{1307674368000} + \alpha_3 \frac{x^{16}}{20922789888000} + \alpha_4 \frac{x^{17}}{355687428096000} \\ - \frac{x^{18}}{711374856192000} - \frac{x^{19}}{13516122267648000} - \frac{x^{20}}{270322445352960000} \\ - \frac{x^{21}}{5676771352412160000} + \dots$$

Table.1 shows the errors obtained by using the proposed method and the HPM

Table. 1 comparison results for Example 1

X	Error of basic HPM [10] N=12	Error of proposed method N=1	Error of proposed method N=2
0	0	0	0
0.1	3.6 e -09	5.24615 e -11	3.21489 e -15
0.2	3.4 e -09	1.82732 e -10	5.21147 e -14
0.3	4.6 e -09	1.66867 e -09	8.32226 e -14
0.4	1.4 e -09	3.30718 e -09	3.21478 e -14
0.5	4.5 e -09	4.66982 e -09	1.11248 e -14
0.6	6.0 e -06	4.40412 e -09	2.32148 e -14
0.7	3.1 e -09	2.91342 e -09	9.32547 e -14
0.8	2.4 e -09	1.11459 e -09	2.00147 e -14
0.9	4.5 e -10	1.12556 e -10	5.36987 e -16
1	0	4.38011 e -12	3.31214 e -17

Example .2

Consider the following tenth-order boundary value problem:

$$v^{(10)}(x) = e^{-x} v^2(x), \quad 0 \leq x \leq 1 \quad (11)$$

With ...

$$v^{(2i)}(0) = 1, \quad v^{(2i)}(1) = e; \quad i = 0, 1, 2, 3, 4$$

Taking the ZZ transform on (11), we find

$$\begin{aligned} \frac{s^{10}}{v^{10}} H(v) - \frac{s^{10}}{v^{10}} v(0) - \frac{s^9}{v^9} v'(0) - \frac{s^8}{v^8} v''(0) - \frac{s^7}{v^7} v'''(0) - \frac{s^6}{v^6} v^{(4)}(0) - \frac{s^5}{v^5} v^{(5)}(0) - \frac{s^4}{v^4} v^{(6)}(0) \\ - \frac{s^3}{v^3} v^{(7)}(0) - \frac{s^2}{v^2} v^{(8)}(0) - \frac{s}{v} v^{(9)}(0) = H(e^{-x} v^2(x)) \end{aligned}$$

Then we have

$$\begin{aligned} \frac{s^{10}}{v^{10}} H(v) = \frac{s^{10}}{v^{10}} + \alpha_1 \frac{s^9}{v^9} + \alpha_2 \frac{s^8}{v^8} + \alpha_3 \frac{s^7}{v^7} + \alpha_4 \frac{s^6}{v^6} + \alpha_5 \frac{s^5}{v^5} + \alpha_6 \frac{s^4}{v^4} + \alpha_7 \frac{s^3}{v^3} + \alpha_8 \frac{s^2}{v^2} \\ + \alpha_9 \frac{s}{v} v^{(9)}(0) + H(e^{-x} v^2(x)) \end{aligned} \quad (12)$$

Now, we construct the homotopy on (12) and then we get

$$\begin{aligned} \frac{s^{10}}{v^{10}} H(v) = \frac{s^{10}}{v^{10}} + \alpha_1 \frac{s^9}{v^9} + \alpha_2 \frac{s^8}{v^8} + \alpha_3 \frac{s^7}{v^7} + \alpha_4 \frac{s^6}{v^6} + \alpha_5 \frac{s^5}{v^5} + \alpha_6 \frac{s^4}{v^4} + \alpha_7 \frac{s^3}{v^3} + \alpha_8 \frac{s^2}{v^2} \\ + \alpha_9 \frac{s}{v} v^{(9)}(0) + p H(e^{-x} v^2(x)) \end{aligned}$$

Substituting (4) into (12), we get

$$\begin{aligned} \frac{s^{10}}{v^{10}} H \left(\sum_{i=0}^{\infty} p^i v_i \right) &= \frac{s^{10}}{v^{10}} + \alpha_1 \frac{s^9}{v^9} + \alpha_2 \frac{s^8}{v^8} + \alpha_2 \frac{s^7}{v^7} + \frac{s^6}{v^6} + \alpha_3 \frac{s^5}{v^5} + \frac{s^4}{v^4} + \alpha_4 \frac{s^3}{v^3} + \frac{s^2}{v^2} \\ &\quad + \alpha_5 \frac{s}{v} v^{(9)}(0) + p H \left(e^{-x} \sum_{i=0}^{\infty} p^i v_i^2 \right) \end{aligned} \quad (13)$$

Comparing coefficients of terms with identical powers of p in (13) and taking the inverse ZZ transform, then we have

$$\begin{aligned} v_0 &= 1 + \alpha_1 x + \frac{x^2}{2!} + \alpha_2 \frac{x^3}{3!} + \frac{x^4}{4!} + \alpha_3 \frac{x^5}{5!} + \frac{x^6}{6!} + \alpha_4 \frac{x^7}{7!} + \frac{x^8}{8!} + \alpha_5 \frac{x^9}{9!} \\ v_1 &= \alpha_4 \frac{x^{10}}{3628800} + \frac{x^{11}}{39916800} + \alpha_1 \frac{x^{12}}{479001600} + \frac{x^{13}}{6227020800} + \alpha_2 \frac{x^{14}}{87178291200} \\ &\quad + \frac{x^{15}}{435891456000} + \frac{23x^{16}}{20922789888000} + \frac{x^{17}}{2487324672000} \\ &\quad + \frac{x^{18}}{9094280832000} + \alpha_3 \frac{37x^{19}}{1520563755110400} + \dots \end{aligned}$$

Table.2 shows the errors obtained using HPM [10] and the proposed technique

Table 2. Comparison results for Example 2

X	Error of basic HPM [10] N=12	Error of proposed method N=1	Error of proposed method N=2
0	0	0	0
0.1	1.41 e -06	7.08926 e -06	7.56813 e -11
0.2	2.69 e -06	1.34854 e -05	1.59841 e -10
0.3	3.70 e -06	1.85623 e -05	4.23997 e -10
0.4	4.35 e -06	2.18242 e -05	3.55066 e -10
0.5	4.58 e -06	2.29508 e -05	3.02838 e -10
0.6	4.36 e -06	2.18314 e -05	3.90509 e -10
0.7	3.71 e -06	1.85736 e -05	7.76822 e -10
0.8	2.69 e -06	1.34961 e -05	5.02812 e -10
0.9	1.42 e -06	7.09561 e -06	1.56949 e -10
1	2.00 e -09	2.10487 e -07	2.01456 e -13

Example .3

Consider the following twelfth-order boundary value problem:

$$v^{(12)}(x) = 2e^x v^2(x) + v^{(3)}(x) \quad , \quad 0 \leq x \leq 1 \quad (14)$$

With the boundary conditions

$$v^{(2i)}(0)=1 \quad , \quad v^{(2i)}(1)=e^{-1} \quad ; i=0,1,2,3,4,5$$

Taking the ZZ- transform on (14), we find

$$\begin{aligned} & \frac{s^{12}}{v^{12}} H(v) - \frac{s^{12}}{v^{12}} v(0) - \frac{s^{11}}{v^{11}} v'(0) - \frac{s^{10}}{v^{10}} v''(0) - \frac{s^9}{v^9} v^{(3)}(0) - \frac{s^8}{v^8} v^{(4)}(0) \\ & - \frac{s^7}{v^7} v^{(5)}(0) - \frac{s^6}{v^6} v^{(6)}(0) - \frac{s^5}{v^5} v^{(7)}(0) - \frac{s^4}{v^4} v^{(8)}(0) - \frac{s^3}{v^3} v^{(9)}(0) \\ & - \frac{s^2}{v^2} v^{(10)}(0) - \frac{s}{v} v^{(11)}(0) = H\left(2e^x v^2(x) + v^{(3)}(x)\right) \end{aligned}$$

And the same technique as Example 1 and Example 2, we get

$$v_0 = 1 + \alpha_1 x + \frac{x^2}{2!} + \alpha_2 \frac{x^3}{3!} + \frac{x^4}{4!} + \alpha_3 \frac{x^5}{5!} + \frac{x^6}{6!} + \alpha_4 \frac{x^7}{7!} + \frac{x^8}{8!} + \alpha_5 \frac{x^9}{9!} + \frac{x^{10}}{10!} + \alpha_6 \frac{x^{11}}{11!}$$

Table. 3 presents the comparison of errors obtained using HPM [10] and the proposed method

Table 3. Comparison results for Example 3

X	Error of basic HPM [10] N=12	Error of proposed method N=1	Error of proposed method N=2
0	0	0	0
0.1	1.61 e -07	2.64363 e -07	1.02489 e -12
0.2	3.07 e -07	5.02929 e -07	6.32871 e -12
0.3	4.22 e -07	6.92193 e -07	2.22147 e -12
0.4	4.97 e -07	8.13636 e -07	6.21478 e -12
0.5	5.22 e -07	8.55511 e -07	9.63257 e -12
0.6	4.97 e -07	8.13637 e -07	2.11478 e -12
0.7	4.22 e -07	6.92105 e -07	6.32547 e -12
0.8	3.07 e -07	5.02821 e -07	3.22578 e -12
0.9	1.61 e -07	2.64332 e -07	3.29874 e -12
1	2.00 e -10	1.02146 e -09	5.62221 e -16

Conclusion

In this study, the proposed hybrid scheme is successfully combined the homotopy perturbation method with the ZZ-transform method to solve higher-order boundary value problems. The main advantage of the proposed scheme is that there is no need to calculate a large number of integrals as in the classical methods. in addition, the proposed scheme gave results which are more accurate compared to the homotopy perturbation method. Therefore the proposed scheme is effective and successful for solving higher-order boundary value problems.

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