

# INJECTIVE EDGE COLORING OF CUBIC GRAPHS

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## **Abstract**

Three edges  $e_1, e_2$  and  $e_3$  in a graph  $G$  are consecutive if they form a cycle of length 3 or a path in this order. A  $k$ -injective edge-coloring of a graph  $G$  is an edge-coloring of  $G$ , (not necessarily proper), such that if edges  $e_1, e_2, e_3$  are consecutive, then  $e_1$  and  $e_3$  receive distinct colors. The minimum  $k$  for which  $G$  has a  $k$ -injective edge-coloring is called the injective edge-coloring number, denoted by  $\chi'_i(G)$ . In this paper, injective edge-coloring numbers of  $H$ -graph and generalized  $H$ -graph are determined.

**Keywords:** *Edge-coloring;  $k$ -injective edge-coloring; injective edge-coloring number;  $H$ -graph and generalized  $H$ -graph.*

## **1. Introduction**

The terminology and notations we refer to Bondy and Murthy [2]. Let  $G$  be a finite, simple, undirected and connected graph. Let  $\Delta(G)$  denote the maximum degree of  $G$ . A *proper vertex (edge) coloring* is a mapping from the vertex (edge) set to a finite set of colors, such that adjacent vertices (edges) receive distinct colors. A  $k$ -injective coloring of a graph  $G$  is a mapping  $\psi : V(G) \rightarrow \{1, 2, \dots, k\}$  such that if two vertices have a common neighbor, then they receive distinct colors. The injective chromatic number of  $G$ , denoted by  $\chi_i(G)$ , is the minimum  $k$  for which  $G$  has a  $k$ -injective coloring. The injective coloring of graphs was originated from the Complexity Theory on Random Access Machines, which was proposed by Hahn et al. [9] and applied to the theory of error correcting codes and the designing of computer networks [1]. Similarly, Cardoso et al. [6] introduced the concept of injective edge-coloring, motivated by a Packet Radio Network problem. Three edges  $e_1, e_2$  and  $e_3$  in a graph  $G$  are consecutive if they form a cycle of length 3 or a path in this order. A  $k$ -injective edge-coloring of a graph  $G$  is a mapping  $\psi : E(G) \rightarrow \{1, 2, \dots, k\}$ , such that if  $e_1, e_2, e_3$  are consecutive, then  $\psi(e_1) \neq \psi(e_3)$ . If there is a  $k$ -injective edge-coloring of  $G$ , then we say that  $G$  is  $k$ -injective edge-colored. The minimum  $k$  for which  $G$  has a  $k$ -injective edge-coloring is called the injective edge-coloring number of  $G$ , denoted by

$\chi'_i(G)$ . Cardoso et al. [6] showed that it is NP-complete to decide whether  $\chi'_i(G) = k$ . They determined the injective edge-coloring numbers for paths, cycles, complete bipartite graphs, and Petersen graph and they also gave bounds on some other classes of graphs.

### Proposition 1.1. [6]

Let  $P_n(C_n)$  be a path (cycle) of order  $n$ ,  $K_{p,q}$  be a complete bipartite graph, and  $P$  be the Petersen graph. Then

1.  $\chi'_i(P_n) = 2$ , for  $n \geq 4$ .
2.  $\chi'_i(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4} \\ 3 & \text{otherwise} \end{cases}$
3.  $\chi'_i(K_{p,q}) = \min\{p, q\}$ .
4.  $\chi'_i(P) = 5$ .

A graph  $G$  is an  $\omega'$  edge injective colorable (*perfect EIC*-) graph if  $\chi'_i(G) = \omega'(G)$ , where  $\omega'(G)$  is the number of edges in a maximum clique of  $G$ . In [11], Yue et al. constructed some perfect EIC-graphs, and gave a sharp bound of the injective edge-coloring number of a 2-connected graph with some forbidden conditions. Bu and Qi [5] and Ferdjallah [8] studied the injective edge coloring of sparse graphs in terms of the maximum average degree. Kostochka [10] studied the injective edge-coloring in terms of the maximum degree. Recently, in [3,4], Bu et al. presented some results on the injective edge-coloring numbers of planar graphs. In this paper, we will consider the injective edge-coloring of  $H$ -graph and generalized  $H$ -graphs.

### 1. Injective edge-coloring of $H$ - graph.

In this section, injective edge-coloring number of  $H$ -graph will be discussed.

[7] The  $H$ -graph  $H(r)$ ,  $r \geq 2$ , is the 3-regular graph of order  $6r$ , with vertex set

$$V(H(r)) = \{u_i, v_i, w_i \mid 0 \leq i \leq 2r - 1\}$$

and edge set (subscripts are taken modulo  $2r$ )

$$\begin{aligned} E(H(r)) = & \{(u_i, u_{i+1}), (w_i, w_{i+1}), (u_i, v_i), (v_i, w_i) \mid 0 \leq i \leq 2r - 1\} \\ & \cup \{(v_{2i}, v_{2i+1}) \mid 0 \leq i \leq r - 1\} \end{aligned}$$

**Theorem 1.1.** If  $r \geq 2$ , Then  $3 \leq \chi'_i(H(r)) \leq 4$ ,

**Proof.**

We consider two cases.

**Case 1. If  $r$  is even,  $r \geq 2$ .**

We define  $\psi : E(H(r)) \rightarrow \{1,2,3\}$  as follows:

$$\psi(u_0u_1) = 1 = \psi(u_0u_{2r-1})$$

$$\text{For } i \in \{1,2, \dots, 2r-2\}, \psi(u_iu_{i+1}) = \begin{cases} 2 & \text{if } i \equiv 1,2 \pmod{4} \\ 1 & \text{if } i \equiv 3,0 \pmod{4} \end{cases}$$

$$\text{For } i \in \{0,1,2, \dots, r-1\}, \psi(u_{2i}u_{2i+1}) = 3,$$

$$\text{For } i \in \{0,1,2, \dots, 2r-1\}, \psi(u_iv_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{4} \\ 3 & \text{if } i \equiv 1,3 \pmod{4} \\ 2 & \text{if } i \equiv 2 \pmod{4} \end{cases}$$

$$\text{For } i \in \{0,1,2, \dots, 2r-1\}, \psi(v_iw_i) = \begin{cases} 2 & \text{if } i \equiv 0 \pmod{4} \\ 3 & \text{if } i \equiv 1,3 \pmod{4} \\ 1 & \text{if } i \equiv 2 \pmod{4} \end{cases}$$

$$\psi(w_0w_1) = 2 = \psi(w_0w_{2r-1}),$$

$$\text{For } i \in \{1,2, \dots, 2r-2\}, \psi(w_iw_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1,2 \pmod{4} \\ 2 & \text{if } i \equiv 3,0 \pmod{4} \end{cases}$$

It is easy to check that  $\psi$  is injective edge-coloring of  $H(r)$ . Hence,  $\chi'_i(H(r)) = 3$ .

**Case 2. If  $r$  is odd,  $r \geq 3$ .**

We define  $\psi : E(H(r)) \rightarrow \{1,2,3,4\}$  as follows:

$$\psi(u_0u_1) = 1 = \psi(u_0u_{2r-1})$$

$$\text{For } i \in \{1,2, \dots, 2r-4\}, \psi(u_iu_{i+1}) = \begin{cases} 2 & \text{if } i \equiv 1,2 \pmod{4} \\ 1 & \text{if } i \equiv 3,0 \pmod{4} \end{cases}$$

$$\psi(u_iu_{i+1}) = 4, \text{ if } i = 2r-3, 2r-2,$$

$$\text{For } i \in \{0,1,2, \dots, r-1\}, \psi(u_{2i}u_{2i+1}) = 3,$$

$$\text{For } i \in \{0,1,2, \dots, 2r-3\}, \psi(u_i v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{4} \\ 3 & \text{if } i \equiv 1,3 \pmod{4} \\ 2 & \text{if } i \equiv 2 \pmod{4} \end{cases}$$

$$\psi(u_i v_i) = \begin{cases} 4 & \text{if } i = 2r-2 \\ 3 & \text{if } i = 2r-1 \end{cases}$$

$$\text{For } i \in \{0,1,2, \dots, 2r-5\}, \psi(v_i w_i) = \begin{cases} 2 & \text{if } i \equiv 0 \pmod{4} \\ 3 & \text{if } i \equiv 1,3 \pmod{4} \\ 1 & \text{if } i \equiv 2 \pmod{4} \end{cases}$$

$$\psi(v_i w_i) = \begin{cases} 4 & \text{if } i = 2r-4 \\ 3 & \text{if } i = 2r-3, 2r-1 \\ 1 & \text{if } i = 2r-2 \end{cases}$$

$$\psi(w_0 w_1) = 2 = \psi(w_0 w_{2r-1})$$

$$\text{For } i \in \{1,2, \dots, 2r-6\}, \psi(w_i w_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1,2 \pmod{4} \\ 2 & \text{if } i \equiv 3,0 \pmod{4} \end{cases}$$

$$\psi(w_i w_{i+1}) = \begin{cases} 4 & \text{if } i = 2r-5, 2r-4 \\ 1 & \text{if } i = 2r-3, 2r-2 \end{cases}$$

It is easy to check that  $\psi$  is injective edge-coloring of  $H(r)$ . Hence,  $\chi'_i(H(r)) = 4$ .

## 2. Injective edge coloring of Generalized $H$ -graphs.

In this section, we study the injective edge-coloring number of generalized  $H$ -graph.

We now consider a natural extension of  $H$ -graphs. For every integer  $r \geq 2$ , the generalised  $H$ -graph  $H^l(r)$  with  $l$  levels,  $l \geq 1$ , is the 3-regular graph of order  $2r(l+2)$ , with vertex set

$$V(H^l(r)) = \{u_j^i / 0 \leq i \leq l+1, 0 \leq j \leq 2r-1\}.$$

And edge set (subscripts are taken modulo  $2r$ )

$$E(H^l(r)) = \{(u_j^0 u_{j+1}^0), (u_j^{l+1} u_{j+1}^{l+1}) / 0 \leq j \leq r-1\} \cup \{u_{2j}^i, u_{2j+1}^i / 1 \leq i \leq l, 0 \leq j \leq r-1\}$$

$$\cup \{u_j^i, u_{j+1}^i / 1 \leq i \leq l, 0 \leq j \leq 2r-1\}$$

**Theorem 2.1.** If  $r \geq 3$  and  $l \geq 2$ , then  $3 \leq \chi'_i(H^l(r)) \leq 4$ .

*Proof.*

We consider six cases and in each case, we first define  $\psi : E(H^l(r)) \rightarrow \{1,2,3,4\}$  as follows:

$$\psi(u_0^0 u_1^0) = 1,$$

$$\text{For } j \in \{1,2, \dots, 2r-5\}, \psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j \equiv 1,2 \pmod{4} \\ 1 & \text{if } j \equiv 3,0 \pmod{4} \end{cases}$$

$$\psi(u_0^0 u_{2r-1}^0) = 1,$$

For  $i \in \{1,2, \dots, l-1\}$  and  $j \in \{0,1,2, \dots, r-3\}$

$$\text{If } i \equiv 1 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = 3, \text{ for } j \in \{0,1,2, \dots, r-3\}$$

$$\text{If } i \equiv 2 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } j \in \{0,2,4, \dots, r-4\} \\ 1 & \text{if } j \in \{1,3,5, \dots, r-3\} \end{cases}$$

$$\text{If } i \equiv 0 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } j \in \{0,2,4, \dots, r-4\} \\ 2 & \text{if } j \in \{1,3,5, \dots, r-3\} \end{cases}$$

$$\text{For } j \in \{0,1,2, \dots, 2r-5\}, \psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{4} \\ 3 & \text{if } j \equiv 1,3 \pmod{4} \\ 2 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

For  $i \in \{1,2, \dots, l-2\}$  and  $j \in \{0,1,2, \dots, 2r-5\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j \equiv 0 \pmod{4} \\ 3 & \text{if } j \equiv 1,3 \pmod{4} \\ 1 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j \equiv 0,3 \pmod{4} \\ 1 & \text{if } j \equiv 1,2 \pmod{4} \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j \equiv 0,2 \pmod{4} \\ 1 & \text{if } j \equiv 1 \pmod{4} \\ 2 & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j \equiv 0,2 \pmod{4} \\ 2 & \text{if } j \equiv 1 \pmod{4} \\ 1 & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j \equiv 0,3 \pmod{4} \\ 2 & \text{if } j \equiv 1,2 \pmod{4} \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{4} \\ 3 & \text{if } j \equiv 1,3 \pmod{4} \\ 2 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

**Case 1. If  $l \equiv 3 \pmod{6}$**

$$\psi(u_{2j}^l u_{2j+1}^l) = 4, \text{ for } j \in \{0,1,2, \dots, r-3\}$$

$$\psi(u_0^{l+1} u_1^{l+1}) = 1, \psi(u_0^{l+1} u_{2r-1}^{l+1}) = 1$$

For  $j \in \{0,1,2, \dots, 2r-5\}$ ,

$$\psi(u_j^{l-1} u_j^l) = \begin{cases} 2 & \text{if } j \equiv 0 \pmod{4} \\ 4 & \text{if } j \equiv 1,3 \pmod{4} \\ 1 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{4} \\ 4 & \text{if } j \equiv 1,3 \pmod{4} \\ 2 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j \equiv 1,2 \pmod{4} \\ 1 & \text{if } j \equiv 3,0 \pmod{4} \end{cases}$$

**If r is odd**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j = 2r-5, 2r-4 \\ 4 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 4 & \text{if } j = 2r-2 \end{cases}$$

If  $j \in \{r-2, r-1\}$  and  $i \in \{1,2, \dots, l-1\}$

$$\psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\psi(u_{2j}^l u_{2j+1}^l) = 4, \text{ if } j \in \{r-2, r-1\}$$

For  $i \in \{1,2, \dots, l-2\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-2 \\ 3 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-2 \\ 3 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\psi(u_j^{l-1} u_j^l) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-2 \\ 4 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 4 & \text{if } j = 2r-3, 2r-1 \\ 3 & \text{if } j = 2r-2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r-5, 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

### If r is even

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r-5, 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 2 & \text{if } j = 2r-2 \end{cases}$$

For  $i \in \{1, 2, \dots, l-1\}$

$$\text{If } j = r-2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r-1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\psi(u_{2j}^l u_{2j+1}^l) = 4, \text{ if } j \in \{r-2, r-1\}$$

For  $i \in \{1, 2, \dots, l-2\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 1 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3 \\ 1 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-1 \\ 2 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 2 & \text{if } j = 2r-2 \end{cases}$$

$$\psi(u_j^{l-1} u_j^l) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 4 & \text{if } j = 2r-3, 2r-1 \\ 1 & \text{if } j = 2r-2 \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 4 & \text{if } j = 2r-3, 2r-1 \\ 2 & \text{if } j = 2r-2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 1 & \text{if } j = 2r-5, 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence,  $\chi'_i(H^l(r)) = 4$ .

### Case 2. If $l \equiv 4 \pmod{6}$

$$\psi(u_{2j}^l u_{2j+1}^l) = 3, \text{ for } j \in \{0, 1, 2, \dots, r-3\},$$

$$\psi(u_0^{l+1} u_1^{l+1}) = 2, \psi(u_0^{l+1} u_{2r-1}^{l+1}) = 1,$$

For  $j \in \{0,1,2, \dots, 2r-5\}$ ,

$$\psi(u_j^{l+1}u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j \equiv 1, 0 \pmod{4} \\ 1 & \text{if } j \equiv 2, 3 \pmod{4} \end{cases}$$

$$\psi(u_j^{l-1}u_j^l) = \begin{cases} 3 & \text{if } j \equiv 0, 2 \pmod{4} \\ 1 & \text{if } j \equiv 1 \pmod{4} \\ 2 & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 3 & \text{if } j \equiv 0, 2 \pmod{4} \\ 2 & \text{if } j \equiv 1 \pmod{4} \\ 1 & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

**If r is odd**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j = 2r-5, 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3 \\ 3 & \text{if } j = 2r-2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r-2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 3 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r-1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 3 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4 \\ 1 & \text{if } j = 2r-3, 2r-2 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-1 \\ 2 & \text{if } j = 2r-3 \\ 1 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \\ 3 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 1 \\ 3 & \text{if } j = 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 2 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 1 \\ 1 & \text{if } j = 2r - 3 \\ 3 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 3 \\ 1 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 3$ .

### If r is even

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r - 2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r - 1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 1 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 1 \\ 1 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 2 \\ 1 & \text{if } j = 2r - 3 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 2 \\ 2 & \text{if } j = 2r - 3 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 1 \\ 2 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 3 \\ 1 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 3$ .

### **Case 3. If $l \equiv 5 \pmod{6}$**

$$\psi(u_{2j}^l u_{2j+1}^l) = 4, \text{ for } j \in \{0, 1, 2, \dots, r - 3\},$$

$$\psi(u_0^{l+1} u_1^{l+1}) = 1,$$

For  $j \in \{0, 1, 2, \dots, 2r - 5\}$ ,

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j \equiv 1, 2 \pmod{4} \\ 1 & \text{if } j \equiv 3, 0 \pmod{4} \end{cases}$$

$$\psi(u_j^{l-1} u_j^l) = \begin{cases} 3 & \text{if } j \in \{0, 2, 4, \dots, 2r - 6\} \\ 4 & \text{if } j \in \{1, 3, 5, \dots, 2r - 5\} \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{4} \\ 4 & \text{if } j \equiv 1, 3 \pmod{4} \\ 2 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

### **If r is odd**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 4 & \text{if } j = 2r - 2 \end{cases}$$

If  $j \in \{r - 2, r - 1\}$  and  $i \in \{1, 2, \dots, l\}$

$$\psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l - 1\}$ , similar to case  $l \equiv 3 \pmod{6}$  and  $r$  is odd.

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 1 & \text{if } j = 2r - 3, 2r - 1 \\ 4 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 2.$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

### If r is even

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 4 & \text{if } j = 2r - 2 \end{cases}$$

For  $i \in \{1, 2, \dots, l - 1\}$

$$\text{If } j = r - 2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r - 1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\psi(u_{2j}^l u_{2j+1}^l) = 4, \text{ if } j \in \{r - 2, r - 1\}$$

For  $i \in \{1, 2, \dots, l - 2\}$ , similar to case  $l \equiv 3 \pmod{6}$  and  $r$  is even.

$$\psi(u_j^{l-1}u_j^l) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 2 \\ 4 & \text{if } j = 2r - 3, 2r - 1 \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 1 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1}u_{j+1}^{l+1}) = \begin{cases} 1 & \text{if } j = 2r - 5, 2r - 4 \\ 2 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_0^{l+1}u_{2r-1}^{l+1}) = 1$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

**Case 4. If  $l \equiv 1 \pmod{6}$  and  $l \neq 1$**

$$\psi(u_{2j}^l u_{2j+1}^l) = 3, \text{ for } j \in \{0, 1, 2, \dots, r-3\},$$

$$\psi(u_0^{l+1}u_1^{l+1}) = 2, \psi(u_0^{l+1}u_{2r-1}^{l+1}) = 2,$$

For  $j \in \{0, 1, 2, \dots, 2r-5\}$ ,

$$\psi(u_j^{l+1}u_{j+1}^{l+1}) = \begin{cases} 1 & \text{if } j \equiv 1, 2 \pmod{4} \\ 2 & \text{if } j \equiv 3, 0 \pmod{4} \end{cases}$$

$$\psi(u_j^{l-1}u_j^l) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{4} \\ 3 & \text{if } j \equiv 1, 3 \pmod{4} \\ 2 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 2 & \text{if } j \equiv 0 \pmod{4} \\ 3 & \text{if } j \equiv 1, 3 \pmod{4} \\ 1 & \text{if } j \equiv 2 \pmod{4} \end{cases}$$

**If r is odd**

For  $i \in \{1, 2, \dots, l-2\}$ , similar to case  $l \equiv 3 \pmod{6}$  and  $r$  is odd.

$$\psi(u_j^{l-1}u_j^l) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 2 \\ 3 & \text{if } j = 2r - 3 \\ 4 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 4 & \text{if } j = 2r - 4, 2r - 1 \\ 3 & \text{if } j = 2r - 3 \\ 1 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1}u_{j+1}^{l+1}) = \begin{cases} 4 & \text{if } j = 2r-5, 2r-4 \\ 1 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

**If r is even**

$$\psi(u_j^0u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r-5, 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0u_j^1) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 2 & \text{if } j = 2r-2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r-2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r-1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 1 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3 \\ 1 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-1 \\ 2 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 1 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 5, 2r - 4 \\ 1 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 3$ .

**If  $l \equiv 2 \pmod{6}$  and  $l \equiv 0 \pmod{6}$**

$$\psi(u_0^0 u_1^0) = 1, \text{ For } j \in \{1, 2, \dots, 2r - 5\}, \psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j \equiv 1, 2 \pmod{4} \\ 3 & \text{if } j \equiv 3, 4 \pmod{4} \\ 1 & \text{if } j \equiv 5, 0 \pmod{4} \end{cases}$$

$$\psi(u_0^0 u_{2r-1}^0) = 1,$$

For  $i \in \{1, 2, \dots, l\}$  and  $j \in \{0, 1, 2, \dots, r - 3\}$

$$\text{If } i \equiv 1 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } j \equiv 0 \pmod{3} \\ 1 & \text{if } j \equiv 1 \pmod{3} \\ 2 & \text{if } j \equiv 2 \pmod{3} \end{cases}$$

$$\text{If } i \equiv 2 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } j \equiv 0 \pmod{3} \\ 3 & \text{if } j \equiv 1 \pmod{3} \\ 1 & \text{if } j \equiv 2 \pmod{3} \end{cases}$$

$$\text{If } i \equiv 0 \pmod{3}, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{3} \\ 2 & \text{if } j \equiv 1 \pmod{3} \\ 3 & \text{if } j \equiv 2 \pmod{3} \end{cases}$$

$$\text{For } j \in \{0, 1, 2, \dots, 2r - 5\}, \psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod{3} \\ 3 & \text{if } j \equiv 1 \pmod{3} \\ 2 & \text{if } j \equiv 2 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$  and  $j \in \{0, 1, 2, \dots, 2r - 5\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j \equiv 0, 5 \pmod{6} \\ 3 & \text{if } j \equiv 1, 2 \pmod{6} \\ 1 & \text{if } j \equiv 3, 4 \pmod{6} \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j \equiv 0, 3 \pmod{6} \\ 1 & \text{if } j \equiv 1, 4 \pmod{6} \\ 3 & \text{if } j \equiv 2, 5 \pmod{6} \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j \equiv 0,5 \pmod{6} \\ 1 & \text{if } j \equiv 1,2 \pmod{6} \\ 2 & \text{if } j \equiv 3,4 \pmod{6} \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j \equiv 0,3 \pmod{6} \\ 2 & \text{if } j \equiv 1,4 \pmod{6} \\ 1 & \text{if } j \equiv 2,5 \pmod{6} \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j \equiv 0,5 \pmod{6} \\ 2 & \text{if } j \equiv 1,2 \pmod{6} \\ 3 & \text{if } j \equiv 3,4 \pmod{6} \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j \equiv 0,3 \pmod{6} \\ 3 & \text{if } j \equiv 1,4 \pmod{6} \\ 2 & \text{if } j \equiv 2,5 \pmod{6} \end{cases}$$

### **Case 5. If $l \equiv 2 \pmod{6}$**

$$\psi(u_0^{l+1} u_1^{l+1}) = 1,$$

$$\text{For } j \in \{1,2, \dots, 2r-5\}, \psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 1 & \text{if } j \equiv 1,0 \pmod{6} \\ 2 & \text{if } j \equiv 2,3 \pmod{6} \\ 3 & \text{if } j \equiv 4,5 \pmod{6} \end{cases}$$

#### **Sub Case 5.1. If $r \equiv 0 \pmod{3}$**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 2 & \text{if } j = 2r-5, 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3 \\ 3 & \text{if } j = 2r-2 \end{cases}$$

For  $i \in \{1,2, \dots, l\}$

$$\text{If } j = r-2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 3 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r-1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 3 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{0,1,2, \dots, l\}$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 1 \\ 1 & \text{if } j = 2r - 3 \\ 3 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4 \\ 1 & \text{if } j = 2r - 3, 2r - 2 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 1 \\ 2 & \text{if } j = 2r - 3 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 2 & \text{if } j = 2r - 3, 2r - 2 \\ 3 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 1 \\ 3 & \text{if } j = 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 2 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 3, \psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 3 \\ 3 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 3$ .

### **Sub Case 5.2. If $r \equiv 1 \pmod{3}$**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 3 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 1 \\ 2 & \text{if } j = 2r - 3 \\ 4 & \text{if } j = 2r - 2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r - 2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 3 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r - 1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l-1\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \\ 3 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-1 \\ 3 & \text{if } j = 2r-3 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-2 \\ 1 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3 \\ 3 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4 \\ 1 & \text{if } j = 2r-3, 2r-2 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-1 \\ 2 & \text{if } j = 2r-3 \\ 1 & \text{if } j = 2r-2 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 4,$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3 \\ 2 & \text{if } j = 2r-2 \\ 4 & \text{if } j = 2r-1 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r-5 \\ 3 & \text{if } j = 2r-4, 2r-3 \\ 4 & \text{if } j = 2r-2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

### **Sub Case 5.3. If $r \equiv 2 \pmod{3}$**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r-5, 2r-4 \\ 4 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-1 \\ 4 & \text{if } j = 2r-2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$  and  $j \in \{r-2, r-1\}$ ,

$$\psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l-1\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-2 \\ 3 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-2 \\ 2 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-2 \\ 3 & \text{if } j = 2r-3, 2r-1 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 4,$$

$$\psi(u_j^l u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-2 \\ 1 & \text{if } j = 2r-3 \\ 4 & \text{if } j = 2r-1 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 3 & \text{if } j = 2r-5 \\ 1 & \text{if } j = 2r-4, 2r-3 \\ 4 & \text{if } j = 2r-2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_l(H^l(r)) = 4$ .

#### **Case 6. If $l \equiv 0 \pmod{6}$**

$$\psi(u_0^{l+1}u_1^{l+1}) = 3$$

$$\text{For } j \in \{1, 2, \dots, 2r-5\}, \psi(u_j^{l+1}u_{j+1}^{l+1}) = \begin{cases} 3 & \text{if } j \equiv 1, 0 \pmod{6} \\ 1 & \text{if } j \equiv 2, 3 \pmod{6} \\ 2 & \text{if } j \equiv 4, 5 \pmod{6} \end{cases}$$

**Sub Case 6.1. If  $r \equiv 0 \pmod{3}$**

$$\psi(u_j^0u_{j+1}^0) = \begin{cases} 2 & \text{if } j = 2r-5, 2r-4 \\ 3 & \text{if } j = 2r-3, 2r-2 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r-2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 3 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r-1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 3 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{0, 1, 2, \dots, l\}$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 2 & \text{if } j = 2r-4, 2r-1 \\ 1 & \text{if } j = 2r-3 \\ 3 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4 \\ 1 & \text{if } j = 2r-3, 2r-2 \\ 2 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 3 & \text{if } j = 2r-4, 2r-1 \\ 2 & \text{if } j = 2r-3 \\ 1 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4 \\ 2 & \text{if } j = 2r-3, 2r-2 \\ 3 & \text{if } j = 2r-1 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_{j+1}^{i+1}) = \begin{cases} 1 & \text{if } j = 2r-4, 2r-1 \\ 3 & \text{if } j = 2r-3 \\ 2 & \text{if } j = 2r-2 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 2 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 2$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 3 & \text{if } j = 2r - 5 \\ 1 & \text{if } j = 2r - 4, 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 3$ .

### **Sub Case 6.2. If $r \equiv 1 \pmod{3}$**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 3 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 3 & \text{if } j = 2r - 4 \\ 1 & \text{if } j = 2r - 3 \\ 4 & \text{if } j = 2r - 2 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r - 2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 3 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r - 1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 3 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l-1\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 1 \\ 1 & \text{if } j = 2r - 3 \\ 3 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3, 2r - 2 \\ 1 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 1 \\ 3 & \text{if } j = 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 2 & \text{if } j = 2r - 3, 2r - 2 \\ 3 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 1 \\ 2 & \text{if } j = 2r - 3 \\ 1 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4 \\ 1 & \text{if } j = 2r - 3, 2r - 2 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 2, \psi(u_j^l u_j^{l+1}) = \begin{cases} 3 & \text{if } j = 2r - 4 \\ 4 & \text{if } j = 2r - 3 \\ 1 & \text{if } j = 2r - 2 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 1 & \text{if } j = 2r - 5 \\ 4 & \text{if } j = 2r - 4, 2r - 3 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to check that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_l(H^l(r)) = 4$ .

### **Sub Case 6.3. If $r \equiv 2 \pmod{3}$**

$$\psi(u_j^0 u_{j+1}^0) = \begin{cases} 1 & \text{if } j = 2r - 5, 2r - 4 \\ 4 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\psi(u_j^0 u_j^1) = \begin{cases} 1 & \text{if } j = 2r - 4 \\ 3 & \text{if } j = 2r - 3 \\ 4 & \text{if } j = 2r - 2 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

For  $i \in \{1, 2, \dots, l\}$

$$\text{If } j = r - 2, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

$$\text{If } j = r - 1, \psi(u_{2j}^i u_{2j+1}^i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod{3} \\ 3 & \text{if } i \equiv 2 \pmod{3} \\ 1 & \text{if } i \equiv 0 \pmod{3} \end{cases}$$

For  $i \in \{1, 2, \dots, l - 1\}$

$$\text{If } i \equiv 1 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, 2r - 1 \\ 3 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\text{If } i \equiv 2 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 2 & \text{if } j = 2r - 4, \\ 1 & \text{if } j = 2r - 3, 2r - 1 \\ 3 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 3 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4 \\ 1 & \text{if } j = 2r - 3, 2r - 1 \\ 2 & \text{if } j = 2r - 2 \end{cases}$$

$$\text{If } i \equiv 4 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 3 & \text{if } j = 2r - 4, 2r - 1 \\ 2 & \text{if } j = 2r - 3, 2r - 2 \end{cases}$$

$$\text{If } i \equiv 5 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 2 \\ 2 & \text{if } j = 2r - 3 \\ 3 & \text{if } j = 2r - 1 \end{cases}$$

$$\text{If } i \equiv 0 \pmod{6}, \psi(u_j^i u_j^{i+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 2 \\ 3 & \text{if } j = 2r - 3 \\ 2 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_0^{l+1} u_{2r-1}^{l+1}) = 4,$$

$$\psi(u_j^l u_j^{l+1}) = \begin{cases} 1 & \text{if } j = 2r - 4, 2r - 2 \\ 3 & \text{if } j = 2r - 3 \\ 4 & \text{if } j = 2r - 1 \end{cases}$$

$$\psi(u_j^{l+1} u_{j+1}^{l+1}) = \begin{cases} 2 & \text{if } j = 2r - 5 \\ 3 & \text{if } j = 2r - 4, 2r - 3 \\ 4 & \text{if } j = 2r - 2 \end{cases}$$

It is easy to verify that  $\psi$  is injective edge-coloring of  $H^l(r)$ . Hence  $\chi'_i(H^l(r)) = 4$ .

### 3. Conclusion

In this paper, I investigated the injective edge-coloring numbers of  $H$ -graphs and generalized  $H$ -graphs. To derive similar results for other graph families is an open area of research.

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