SOLVING SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS USING A NEW HYBRID SEMI-ANALYTICAL METHOD

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Abstract

This paper presents a new hybrid method that combines Kharrat-Toma transform technique with the homotopy perturbation method for the solutions of linear and nonlinear initial value problems represented by integro-differential equations systems with initial conditions. Where, the advantages of this hybrid approach are rapid convergence to an approximate or exact solution, reduces the computational steps and integrals which making it a highly efficient and applicable method compared with the classical methods. Therefore, this proposed new hybrid approach (Kharrat-Toma transform and homotopy perturbation method) can be apply to find the analytical solutions of integro-differential equations systems arising in many applications. To show the accuracy and effectiveness of the presented technique several examples are introduced.

Keywords*: Homotopy perturbation method, Kharrat-Toma transform, System of Integro-differential equations.*

INTRODUCTION

Integro-differential equations system arises in many fields of engineering, science and mathematical physics applications. Therefore, the researchers were interested in these types of equations, but sometimes it is difficult to solve it by traditional methods, so the researchers and academicians suggested new hybrid methods that are more efficient than classical methods. There are various analytical and numerical methods were used for these types of problems. Bakodah et.al [1] proposed method on the discrete Adomian decomposition method to solve Volterra integro-differential equations and Fredholm integro-differential equations. In [2] the Petrov-Galerkin method has been discussed for solving Integro-differential equations system. Jafarzadeh and Keramati presented numerical method based on Taylor polynomial for solving system of higher order linear integro differential equations [3]. Hesameddini and Rahimi introduced a new numerical scheme includes reconstruction of variational iteration method with Laplace transform to handle the systems of integro-differential equations [4]. Sinc-Derivative collocation method was used in [5] to solve second-order integro-differential boundary value problems.

In [6], Issa et.al, applied a collocation method using sinc functions and Chebyshev wavelet method to approximate the solution of systems of Volterra integro-differential equations. Hamaydi and Qatanania [7] employed the variational iteration method and the Taylor expansion to solve Volterra equation. Biazar and Ebrahimi [8] introduced an extension of Chebeyshev wavelets method to solve integro-differential equations systems. Moreover, many numerical methods were used for integro-differential equations systems, such as single term Walsh series [9], rationalized Haar functions [10], power series [11], differential transform [12], finite difference approximation method [13], homotopy perturbation method [14].

A new hybrid scheme based on Kharrat-Toma transform with homotopy perturbation method is suggested in this paper to find solutions of integro-differential equations systems. The proposed hybrid approach finds an infinite series solution, which convergence to an exact or approximate solution.

The article is ordered as follows: in Section 2, the Kharrat-Toma transform method is presented. The new hybrid method and methodology is outlined in Section 3. In Section 4, the proposed method is implemented for three numerical examples. Finally, conclusions are showed in Section 5.

KHARRAT-TOMA TRANSFORM METHOD:

The new Kharrat-Toma integral transform method is presented by Kharrat and Toma (2020) to solve initial or boundary value problems.

Definition: [15] The Kharrat-Toma integral transform technique of a function $f(x)$ is described as follows:

$$B\left[f(x)\right]=G(S)=s^{3}\int_{0}^{\infty}f(x)\,e^{\frac{-x}{s^{2}}}dx \quad , \quad x\geq 0$$

The inverse Kharrat-Toma integral transform is defined as:

$$f(x)=B^{-1}\left[G(S)\right]=B^{-1}\left[s^{3}\int_{0}^{\infty}f(x)\,e^{\frac{-x}{s^{2}}}dx\right]$$

The B^{-1} will be the inverse of the B integral transform. Where

$$B\left[f^{(n)}(x)\right]=\frac{1}{s^{2n}}G(s)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f^{(k)}(0) \quad ; \; n\geq 1$$

Methodology

The methodology of the proposed hybrid method to solve system of linear and nonlinear integro-differential equations is illustrated in this section.

Consider the system of integro-differential equations as follows:

$$\begin{cases} f_1^{(n)}(x)=\psi_1\left(x,f_2(x),\dots,f_2^{(n)}(x),f_3(x),\dots,f_3^{(n)}(x),f_m(x),\dots,f_m^{(n)}(x)\right)+\int_0^x K_1\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt \\ f_2^{(n)}(x)=\psi_2\left(x,f_1(x),\dots,f_1^{(n)}(x),f_3(x),\dots,f_3^{(n)}(x),f_m(x),\dots,f_m^{(n)}(x)\right)+\int_0^x K_2\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt \\ \vdots \\ f_m^{(n)}(x)=\psi_m\left(x,f_1(x),\dots,f_1^{(n)}(x),f_2(x),\dots,f_2^{(n)}(x),f_{m-1}(x),\dots,f_{m-1}^{(n)}(x)\right)+\int_0^x K_m\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt \end{cases} \tag{1}$$

With initial conditions

$$f_m^{(n)}(0)=\alpha_{n,m} \quad , \quad \alpha_{n,m}=const \quad , \quad n=0,1,\dots,a-1 \quad , \quad m=1,2,\dots,b$$

Taking the Kharrat-Toma transform on (1), yields

$$\begin{cases} \frac{1}{s^{2n}}B(f_1)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f_1^{(k)}(0)=B\left[\psi_1\left(x,f_2(x),\dots,f_2^{(n)}(x),f_3(x),\dots,f_3^{(n)}(x),f_m(x),\dots,f_m^{(n)}(x)\right)\right] \\ \qquad +B\left[\int_0^x K_1\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt\right] \\ \frac{1}{s^{2n}}B(f_2)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f_2^{(k)}(0)=B\left[\psi_2\left(x,f_1(x),\dots,f_1^{(n)}(x),f_3(x),\dots,f_3^{(n)}(x),f_m(x),\dots,f_m^{(n)}(x)\right)\right] \\ \qquad +B\left[\int_0^x K_2\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt\right] \\ \vdots \\ \frac{1}{s^{2n}}B(f_m)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f_m^{(k)}(0)=B\left[\psi_m\left(x,f_1(x),\dots,f_1^{(n)}(x),f_2(x),\dots,f_2^{(n)}(x),f_{m-1}(x),\dots,f_{m-1}^{(n)}(x)\right)\right] \\ \qquad +B\left[\int_0^x K_m\left(x,t,f_1(t),\dots,f_1^{(n)}(t),\dots,f_m(t),\dots,f_m^{(n)}(t)\right)dt\right] \end{cases} \tag{2}$$

Then

$$\begin{cases} B(f_1)=\sum_{k=0}^{n-1} s^{2k+5} f_1^{(k)}(0)+s^{2n}B\left[\psi_1\left(x,f_2(x),\ldots,f_2^{(n)}(x),f_3(x),\ldots,f_3^{(n)}(x),f_m(x),\ldots,f_m^{(n)}(x)\right)\right] \\ \qquad +s^{2n}B\left[\int_0^x K_1\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \\ B(f_2)=\sum_{k=0}^{n-1} s^{2k+5} f_2^{(k)}(0)+s^{2n}B\left[\psi_2\left(x,f_1(x),\ldots,f_1^{(n)}(x),f_3(x),\ldots,f_3^{(n)}(x),f_m(x),\ldots,f_m^{(n)}(x)\right)\right] \\ \qquad +s^{2n}B\left[\int_0^x K_2\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \\ \vdots \\ B(f_m)=\sum_{k=0}^{n-1} s^{2k+5} f_m^{(k)}(0)+s^{2n}B\left[\psi_m\left(x,f_1(x),\ldots,f_1^{(n)}(x),f_2(x),\ldots,f_2^{(n)}(x),f_{m-1}(x),\ldots,f_{m-1}^{(n)}(x)\right)\right] \\ \qquad +s^{2n}B\left[\int_0^x K_m\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \end{cases} \tag{3}$$

The homotopy of (3) can be written as:

$$\begin{cases} B(f_1)=\sum_{k=0}^{n-1} s^{2k+5} f_1^{(k)}(0)+p\,s^{2n}B\left[\psi_1\left(x,f_2(x),\ldots,f_2^{(n)}(x),f_3(x),\ldots,f_3^{(n)}(x),f_m(x),\ldots,f_m^{(n)}(x)\right)\right] \\ \qquad +p\,s^{2n}B\left[\int_0^x K_1\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \\ B(f_2)=\sum_{k=0}^{n-1} s^{2k+5} f_2^{(k)}(0)+p\,s^{2n}B\left[\psi_2\left(x,f_1(x),\ldots,f_1^{(n)}(x),f_3(x),\ldots,f_3^{(n)}(x),f_m(x),\ldots,f_m^{(n)}(x)\right)\right] \\ \qquad +p\,s^{2n}B\left[\int_0^x K_2\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \\ \vdots \\ B(f_m)=\sum_{k=0}^{n-1} s^{2k+5} f_m^{(k)}(0)+p\,s^{2n}B\left[\psi_m\left(x,f_1(x),\ldots,f_1^{(n)}(x),f_2(x),\ldots,f_2^{(n)}(x),f_{m-1}(x),\ldots,f_{m-1}^{(n)}(x)\right)\right] \\ \qquad +p\,s^{2n}B\left[\int_0^x K_m\left(x,t,f_1(t),\ldots,f_1^{(n)}(t),\ldots,f_m(t),\ldots,f_m^{(n)}(t)\right)dt\right] \end{cases} \tag{4}$$

Where $p \in [0,1]$ is an embedding parameter.

According to the homotopy perturbation method HPM the solution of (4) can be written as a power series in p

$$f_m = \sum_{i=0}^{\infty} p^i f_{m_i} \tag{5}$$

Substituting (5) into (4), and comparing the coefficients of terms with identical powers of p and taking the inverse Kharrat-Toma transform, gives

Setting $p = 1$, we get the approximate solution of (1)

$$f_m(x) = \sum_{i=0}^{\infty} f_{m_i}(x)$$

Numerical Examlpes

In this part, we will implement the proposed hybrid approach for three integro-differential equations systems to test the power and efficiency of this new hybrid technique.

Example .1

Consider the nonlinear integro-differential equation system of the form:

$$\begin{cases} u'(x) = 2x + \dfrac{x^4}{6} + \dfrac{2x^6}{15} + \displaystyle\int_0^x (x - 2t)\left(u^2(t) + v(t)\right)dt \\ v'(x) = -2x - \dfrac{x^4}{6} + \dfrac{2x^6}{15} + \displaystyle\int_0^x (x - 2t)\left(u(t) + v^2(t)\right)dt \end{cases} \tag{6}$$

With initial conditions

$$u(0) = 1 \ , \ v(0) = 1$$

Taking the Kharrat-Toma transform on (6), finds

$$\begin{cases} \dfrac{1}{s^2} B(u) - s^3 u(0) = B\left[2x + \dfrac{x^4}{6} + \dfrac{2x^6}{15}\right] + B\left[\displaystyle\int_0^x (x - 2t)\left(u^2(t) + v(t)\right)dt\right] \\ \dfrac{1}{s^2} B(v) - s^3 v(0) = B\left[-2x - \dfrac{x^4}{6} + \dfrac{2x^6}{15}\right] + B\left[\displaystyle\int_0^x (x - 2t)\left(u(t) + v^2(t)\right)dt\right] \end{cases} \tag{7}$$

Then we have

$$\begin{cases} B(u)=s^5+s^2B\left[2x+\frac{x^4}{6}+\frac{2x^6}{15}\right]+s^2B\left[\int_0^x (x-2t)\left(u^2(t)+v(t)\right)dt\right] \\ B(v)=s^5+s^2B\left[-2x-\frac{x^4}{6}+\frac{2x^6}{15}\right]+s^2B\left[\int_0^x (x-2t)\left(u(t)+v^2(t)\right)dt\right] \end{cases} \tag{8}$$

Now, constructing the homotopy on (8) as follows

$$\begin{cases} B(u)=s^5+s^2B\left[2x+\frac{x^4}{6}+\frac{2x^6}{15}\right]+ps^2B\left[\int_0^x (x-2t)\left(u^2(t)+v(t)\right)dt\right] \\ B(v)=s^5+s^2B\left[-2x-\frac{x^4}{6}+\frac{2x^6}{15}\right]+ps^2B\left[\int_0^x (x-2t)\left(u(t)+v^2(t)\right)dt\right] \end{cases} \tag{9}$$

Substituting (5) into (9), we get

$$\begin{cases} B\left(\sum_{i=0}^{\infty}p^i u_i\right)=s^5+s^2B\left[2x+\frac{x^4}{6}+\frac{2x^6}{15}\right]+ps^2B\left[\int_0^x (x-2t)\left(\left(\sum_{i=0}^{\infty}p^i u_i(t)\right)^2+\sum_{i=0}^{\infty}p^i v_i(t)\right)dt\right] \\ B\left(\sum_{i=0}^{\infty}p^i v_i\right)=s^5+s^2B\left[-2x-\frac{x^4}{6}+\frac{2x^6}{15}\right]+ps^2B\left[\int_0^x (x-2t)\left(\sum_{i=0}^{\infty}p^i u_i(t)+\left(\sum_{i=0}^{\infty}p^i v_i(t)\right)^2\right)dt\right] \end{cases} \tag{10}$$

Comparing coefficients of terms with identical powers of p in (10), leads to

$$p^0:\begin{cases} B[u_0]=s^5+s^2B\left[2x+\frac{x^4}{6}+\frac{2x^6}{15}\right] \\ B[v_0]=s^5+s^2B\left[-2x-\frac{x^4}{6}+\frac{2x^6}{15}\right] \end{cases} \tag{11}$$

$$p^1:\begin{cases} B[u_1]=s^2B\left[\int_0^x (x-2t)\left((u_0(t))^2+v_0(t)\right)dt\right] \\ B[v_1]=s^2B\left[\int_0^x (x-2t)\left(u_0(t)+(v_0(t))^2\right)dt\right] \end{cases} \tag{12}$$

Taking the inverse Kharrat-Toma transform of Equations. (11) and (12), yields

$$\begin{cases} u_0 = 1 + x^2 + \dfrac{x^5}{30} + \dfrac{2x^7}{105} \\ v_0 = 1 - x^2 - \dfrac{x^5}{30} + \dfrac{2x^7}{105} \end{cases}$$

$$\begin{cases} u_1 = -\dfrac{x^5}{30} - \dfrac{2x^7}{105} - \dfrac{x^8}{2016} - \dfrac{13x^{10}}{10800} - \dfrac{x^{12}}{3850} - \dfrac{x^{13}}{154440} - \dfrac{4x^{15}}{716625} - \dfrac{x^{17}}{803250} \\ v_0 = \dfrac{x^5}{30} - \dfrac{2x^7}{105} + \dfrac{x^8}{2016} - \dfrac{13x^{10}}{10800} + \dfrac{x^{12}}{3850} - \dfrac{x^{13}}{154440} + \dfrac{4x^{15}}{716625} - \dfrac{x^{17}}{803250} \end{cases}$$

Table 1. shows the comparison of absolute errors results by taking two terms $u_0 + u_1$ and $v_0 + v_1$ of the proposed hybrid method.
Where the exact solution of Eqn. (6) is
$u(x) = 1 + x^2$, $v(x) = 1 - x^2$

TABLE 1: The comparison of absolute errors for Example 1

x	Error of proposed hybrid method (n=2) u_0+u_1	Error of proposed hybrid method (n=2) v_0+v_1
0	0	0
0.1	5.08100 e -12	4.84020 e -12
0.2	1.39417 e -09	1.14763 e -09
0.3	3.97915 e -08	2.55740 e -08
0.4	4.55704 e -07	2.03182 e -07
0.5	3.17750 e -06	8.24916 e -07
0.6	1.61865 e -05	1.61245 e -06
0.7	6.62842 e -05	1.85040 e -06
0.8	2.30897 e -04	2.83648 e -05
0.9	7.09592 e -04	1.33526 e -04
1	1.97278 e -03	4.50070 e -04

Example .2

Consider the system of nonlinear integro-differential equation

$$\begin{cases} u''(x) = 1 - \dfrac{x^3}{3} - \dfrac{1}{2}v'^2(x) + \dfrac{1}{2}\displaystyle\int_0^x \left(u^2(t) + v^2(t)\right)dt \\ v''(x) = -1 + x^2 - x\,u(x) + \dfrac{1}{4}\displaystyle\int_0^x \left(u^2(t) - v^2(t)\right)dt \end{cases} \tag{13}$$

With initial conditions

$u(0) = 1$, $u'(0) = 2,$, $v(0) = -1$, $v'(0) = 0$

Taking the Kharrat-Toma transform on (13), yields

$$\begin{cases} \dfrac{1}{s^4}B(u) - s\,u(0) - s^3 u'(0) = B\left[1 - \dfrac{x^3}{3}\right] + B\left[-\dfrac{1}{2}v'^2(x) + \dfrac{1}{2}\displaystyle\int_0^x \left(u^2(t) + v^2(t)\right)dt\right] \\ \dfrac{1}{s^4}B(v) - s\,v(0) - s^3 v'(0) = B\left[-1 + x^2\right] + B\left[-x\,u(x) + \dfrac{1}{4}\displaystyle\int_0^x \left(u^2(t) - v^2(t)\right)dt\right] \end{cases} \tag{14}$$

Then we have

$$\begin{cases} B(u) = s^5 + 2s^7 + s^4 B\left[1 - \dfrac{x^3}{3}\right] + s^4 B\left[-\dfrac{1}{2}v'^2(x) + \dfrac{1}{2}\displaystyle\int_0^x \left(u^2(t) + v^2(t)\right)dt\right] \\ B(v) = -s^5 + s^4 B\left[-1 + x^2\right] + s^4 B\left[-x\,u(x) + \dfrac{1}{4}\displaystyle\int_0^x \left(u^2(t) - v^2(t)\right)dt\right] \end{cases} \tag{15}$$

Now, constructing the homotopy on (15) as follows

$$\begin{cases} B(u) = s^5 + 2s^7 + p\,s^4 B\left[1 - \dfrac{x^3}{3}\right] + p\,s^4 B\left[-\dfrac{1}{2}v'^2(x) + \dfrac{1}{2}\displaystyle\int_0^x \left(u^2(t) + v^2(t)\right)dt\right] \\ B(v) = -s^5 + p\,s^4 B\left[-1 + x^2\right] + p\,s^4 B\left[-x\,u(x) + \dfrac{1}{4}\displaystyle\int_0^x \left(u^2(t) - v^2(t)\right)dt\right] \end{cases} \tag{16}$$

Substituting (5) into (16), we get

$$\begin{cases} B\left(\displaystyle\sum_{i=0}^{\infty} p^i u_i\right) = s^5 + 2s^7 + p\,s^4 B\left[1 - \dfrac{x^3}{3}\right] + p\,s^4 B\left[-\dfrac{1}{2}\left(\displaystyle\sum_{i=0}^{\infty} p^i v_i'(x)\right)^2 + \dfrac{1}{2}\displaystyle\int_0^x \left(\left(\sum_{i=0}^{\infty} p^i u_i(t)\right)^2 + \left(\sum_{i=0}^{\infty} p^i v_i(t)\right)^2\right)dt\right] \\ B\left(\displaystyle\sum_{i=0}^{\infty} p^i v_i\right) = -s^5 + p\,s^4 B\left[-1 + x^2\right] + p\,s^4 B\left[-x\displaystyle\sum_{i=0}^{\infty} p^i u_i(x) + \dfrac{1}{4}\displaystyle\int_0^x \left(\left(\sum_{i=0}^{\infty} p^i u_i(t)\right)^2 - \left(\sum_{i=0}^{\infty} p^i v_i(t)\right)^2\right)dt\right] \end{cases} \tag{17}$$

Comparing coefficients of terms with identical powers of p in (17) and taking the inverse Kharrat-Toma transform of equations results, yields

$$\begin{cases} u_0 = 1 + 2x + \dfrac{x^2}{2} - \dfrac{x^6}{60} \\ v_0 = -1 - \dfrac{x^2}{2} + \dfrac{x^4}{12} \end{cases}$$

$$\begin{cases} u_1 = \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{20} + \dfrac{7x^6}{360} + \dfrac{x^7}{1260} - \dfrac{x^8}{960} - \dfrac{x^9}{6720} - \dfrac{x^{10}}{86400} + \dfrac{x^{11}}{285120} + \dfrac{x^{13}}{12355200} \\ v_1 = -\dfrac{x^3}{6} - \dfrac{x^4}{8} - \dfrac{x^5}{120} + \dfrac{x^6}{240} + \dfrac{x^7}{5040} + \dfrac{11x^8}{40320} + \dfrac{x^9}{120960} - \dfrac{x^{10}}{172800} - \dfrac{x^{11}}{570240} + \dfrac{x^{13}}{24710400} \end{cases}$$

Table 2. shows the comparison of absolute errors results by taking two terms $u_0 + u_1$ and $v_0 + v_1$ of the proposed hybrid method.

Where the exact solution of Eqn. (13) is

$u(x) = x + e^x$, $v(x) = x - e^x$

TABLE 2: The comparison of absolute errors for Example 2

x	Error of proposed hybrid method (n=2) u_0+u_1	Error of proposed hybrid method (n=2) v_0+v_1
0	0	0
0.1	2.68180 e -07	5.52251 e -09
0.2	9.16053 e -06	3.61242 e -07
0.3	7.39692 e -05	4.15692 e -06
0.4	3.30191 e -04	2.36034 e -05
0.5	1.06355 e -03	9.10839 e -05
0.6	2.78356 e -03	2.75379 e -04
0.7	6.30713 e -03	7.03698 e -04
0.8	1.28500 e -02	1.59025 e -03
0.9	2.41216 e -02	3.27218 e -03
1	4.24245 e -02	6.25383 e -03

Example .3

Consider the system of integro-differential equation

$$\begin{cases} u''(x) = -1 - x + \cosh x - \dfrac{\sin^3 x}{3} - \sinh x + e^x + \displaystyle\int_0^x \left(\left(e^{-t} \right) u(t) + \left(\sin^2 t \right) v(t) \right) dt \\ v''(x) = -3 + x^2 - \dfrac{2x^3}{3} - 2(x-1)e^x + \displaystyle\int_0^x \left(\left(x^2 - t^2 \right) u(t) + (x-t) v(t) \right) dt \end{cases} \tag{18}$$

With initial conditions

$u(0) = 2$, $u'(0) = 1$, , $v(0) = 1$, $v'(0) = 0$

And the same technique as example .1 and .2, we get

$$\begin{cases} u_0 = 2 + x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{12} - \dfrac{x^5}{60} + \dfrac{x^6}{360} + \dfrac{x^7}{252} + \dfrac{x^8}{20160} - \dfrac{13x^9}{25920} + \dfrac{x^{10}}{1814400} + \dfrac{41x^{11}}{997920} \\ v_0 = 1 - \dfrac{x^2}{2} - \dfrac{x^5}{15} - \dfrac{x^6}{120} - \dfrac{x^7}{630} - \dfrac{x^8}{4032} - \dfrac{x^9}{30240} - \dfrac{x^{10}}{259200} - \dfrac{x^{11}}{2494800} \end{cases}$$

$$\begin{cases} u_1 = \dfrac{x^3}{3} - \dfrac{x^4}{24} + \dfrac{x^5}{40} - \dfrac{x^6}{240} - \dfrac{x^7}{504} - \dfrac{29x^8}{40320} + \dfrac{227x^9}{362880} - \dfrac{491x^{10}}{3628800} - \dfrac{563x^{11}}{19958400} + \dfrac{259x^{12}}{17107200} + \dfrac{359x^{13}}{141523200} - \dfrac{51221x^{14}}{43589145600} + \ldots \\ v_1 = \dfrac{x^4}{24} + \dfrac{x^5}{15} + \dfrac{x^6}{144} + \dfrac{x^7}{630} - \dfrac{x^8}{4032} + \dfrac{x^9}{22680} - \dfrac{17x^{10}}{1814400} + \dfrac{x^{11}}{1663200} + \dfrac{x^{12}}{1368576} + \dfrac{x^{13}}{222393600} - \dfrac{41x^{14}}{889574400} + \ldots \end{cases}$$

Table 3. shows the comparison of absolute errors results by taking two terms $u_0 + u_1$ and $v_0 + v_1$ of the proposed hybrid method.

Where the exact solution of Eqn. (18) is

$$u(x) = e^x + 1 \ , \ v(x) = \cos x$$

TABLE 3: The comparison of absolute errors for Example 3

x	Error of proposed hybrid method (n=2) u_0+u_1	Error of proposed hybrid method (n=2) v_0+v_1
0	0	0
0.1	1.86384 e -09	2.71720 e -11
0.2	1.56480 e -07	1.28777 e -09
0.3	1.67907 e -06	3.40049 e -08
0.4	8.88811 e -06	3.39780 e -07
0.5	3.20504 e -05	2.02536 e -06
0.6	9.07880 e -05	8.71294 e -06
0.7	2.18214 e -04	2.99323 e -05
0.8	4.66190 e -04	8.72269 e -05
0.9	9.12492 e -04	2.24188 e -04
1	1.67100 e -03	5.21873 e -04

CONCLUSIONS

In conclusion, a new hybrid method to solve systems of integro-differential equations is suggested. The presented method is depended on the hybridization between Kharrat-Toma transform with the homotopy perturbation method. The new hybrid approach is easy to apply and reduce the computational steps compared to other traditional methods. Thus, the new hybrid method can be applied to solve several integro-differential equations systems arising in many applications.

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