FIXED POINT THEOREMS IN REVISED FUZZY METRIC SPACES

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Abstract

In this paper, we examine the existence of fixed point results for Banach and Edelstein contraction theorems in Revised fuzzy metric spaces with the assistance of Grabiec. Thus, we create a new path in Revised fuzzy theory to obtain fixed point results. We hope that this paper creates a new way to come up with several fixed point results in the Revised fuzzy metric spaces.

Keywords: *Revised fuzzy metric space, continuous t-conorm, banach contraction, Edelstein contraction, fixed point*

INTRODUCTION

Zadeh designed the hypothesis of fuzzy sets in 1965 [11]. The class of elements with grade of membership deals in fuzzy metric space was presented at first by Kramosil and Michalek [7]. Later on, George and Veeramani [3] gave the altered idea of fuzzy metric spaces because of Kramosil and Michalek [7] and examined a Hausdorff geography of fuzzy metric spaces. As of late, Gregori et al. [5] gave many fascinating instances of fuzzy metrics with regards to the feeling of George and Veeramani [3] and have additionally applied these fuzzy measurements to shading picture handling. In 1977 B. E. Rhoades [10] established the various definitions of contractive mappings, which are very import results for the researchers. Later M. Grabiec [5] and Goebel [4] introduce the existence of fixed points in fuzzy metric space in 1988 and existence of fixed point in metric space. As late of many authors have examined the concept of fuzzy metric space in various aspects.

Alexander Sostak [1] described the concept of "George-Veeramani Fuzzy Metrics Revised". Later on Olga Grigorenko [9], Juan jose Minana, Alexander Sostak, Oscar Valero introduced "On t-conorm based Fuzzy (Pseudo) metrics". In 2020, Alexander Sostak and Tarkan Öner [2] initiate the concept of On Metric-Type Spaces supported Extended t-Conorms. In 2021, Muraliraj.M

& Thangathamizh.R [8] firstly introduced the existence of fixed point theorems in Revised fuzzy metric space [8] in 2021.

2. PRELIMINARIES

To initiate the concept of Revised fuzzy metric space, which was introduced by Alexander Sostak [1] in 2018 is recalled here.

Definition 2.1: [1]

A binary operation $\oplus$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-conorm if it satisfies the following conditions:

a) $\oplus$ is associative andcommutative,

b) $\oplus$ is continuous,

c) $a \oplus 0 = a$ for all $a \in [0,1]$,

d) $a \oplus b \leq c \oplus d$ whenever $a \leq c$ and $b \leq d$

for all $a, b, c, d \in [0,1]$.

Examples 2.2: [2]

i.Lukasievicz t-conorm: $a \oplus b = max\{a, b\}$
ii.Product t-conorm: $a \oplus b = a + b - ab$
iii.Minimum t-conorm: $a \oplus b = min(a + b, 1)$

Definition 2.3: [1]

A Revised fuzzy metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a nonempty set, $\oplus$ is a continuous t-conorm and μ is a Revised fuzzy set on $X \times X \times (0, \infty) \rightarrow [0,1]$ satisfies the following conditions:

$\forall x, y, z \in X \; and \; s, t > 0$

(RGV1) $\mu(x, y, t) < 1, \forall t > 0$

(RGV2) $\mu(x, y, t) = 0$ if and only if $x = y, t > 0$

(RGV3) $\mu(x, y, t) = \mu(y, x, t)$

(RGV4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$

(RGV5) $\mu(x, y, -): (0, \infty) \rightarrow [0,1)$ is continuous.

Then μ is called a Revised fuzzy metric on X.

Example 2.4: [1]

Let (X, d) be a metric space. Define $a \oplus b = max\{a, b\}$ for all $a, b \in [0,1]$, and define $\mu : X \times X \times (0, \infty) \rightarrow [0,1]$ as

$$\mu(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

$\forall x, y, z \in X \; and \; t > 0$. Then $(X, \mu, \oplus)$ is a Revised fuzzy metric space.

Definition 2.5: [1]

Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space, for $t > 0$ the open ball $B(x, r, t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : \mu(x, y, t) < r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and

$0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the Revised fuzzy metric μ.

We call this fuzzy metric induced by the metric d as the standard Revised fuzzy metric.

Definition 2.6: [8]

Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space,

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $lim_{n \to \infty} \mu(x, y, t) = 0 \; for \; all \; t > 0$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\,(x_n, x_m, t) < \epsilon$ for each $n, m \geq n_0$
3. A Revised fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
4. A Revised fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Lemma 2.7: [8]
Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space. For all $u, v \in X, \mu(u, v, -)$ is non-increasing function.

MAIN RESULT

Theorem 3.1: (Banach Contraction Theorem in Revised Fuzzy metric)

Let $(\Sigma, \mu, \oplus)$ be a complete Revised fuzzy metric space. Let $F : \Sigma \to \Sigma$ be a function satisfying,

$$\mu(F\zeta, F\eta, \lambda) \leq \mu(\zeta, \eta, \lambda) \tag{3.1}$$

for all $\zeta, \eta \in \Sigma$. $0 < k < 1$. Then z has unique fixed point.

Proof:

Let $\zeta \in \Sigma \; and \; \{\zeta_n\} = f^n(a) \; (n \in \mathbb{N})$.

By Mathematical induction, we obtain

$$\mu(\zeta_n, \zeta_{n+p}, \lambda) \leq \mu(\zeta, \zeta_1, \tfrac{\lambda}{k^n}) \tag{3.2}$$

for all $n > 0$ and $\lambda > 0$. Thus for any non-negative integer p, we have

$$\mu(\zeta_n, \zeta_{n+p}, \lambda) \leq \mu(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \oplus \cdots (p - times) \cdots \oplus \mu(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p})$$

$$\leq \mu(\zeta, \zeta_{n+1}, \frac{\lambda}{pk^n}) \oplus \cdots (p - times) \cdots \oplus \mu(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}})$$

by (3.2) and the definition of Revised fuzzy metric space conditions,

we get

$$lim_{n\to\infty} \mu\left(\zeta_n, \zeta_{n+p}, \lambda\right) \leq 0 \oplus \cdots (p - times) \cdots \oplus 0 = 0$$

Therefore, $\{\zeta_n\}$ is Cauchy sequence and it is convergent to a limit, let the limit point is η. Thus, we get

$$\mu(F\eta, \eta, \lambda) \leq \mu\,(F\eta, F\zeta_n, \tfrac{\lambda}{2}) \oplus \mu\,(\zeta_{n+1}, \eta, \tfrac{\lambda}{2})$$

$$\leq \mu\left(\eta, \zeta_n, \tfrac{\lambda}{2k}\right) \oplus \mu\left(\zeta_{n+1}, \eta, \tfrac{\lambda}{2}\right) \to 0 \oplus 0 = 0$$

Since we see that $\mu(\zeta, \eta, \lambda) = 0 \ iff \ \zeta = \eta$

We get $F\eta = \eta$, which is the fixed point of Revised fuzzy metric space. To show the uniqueness, let us assume that $F\omega = \omega$ for some $\omega \in \Sigma$

$$0 \leq \mu(\zeta, \omega, \lambda) = \mu\,(F\eta, F\omega F\omega, \lambda) \leq \mu\left(\zeta, \omega, \tfrac{\lambda}{K}\right) = \mu(F\zeta, F\omega, \tfrac{\lambda}{K}) \leq \mu(\eta, \omega, \tfrac{\lambda}{K^2})$$

$$\leq \cdots \leq \mu\,(\zeta, \omega, \tfrac{\lambda}{K^n}) \to 0 \ as \ n \to \infty$$

From the definition of Revised fuzzy metric space, We get $\eta = \omega$.

Therefore z has a unique fixed point.

Lemma 3.2:

Let $(\Sigma, \mu, \oplus)$ be a complete Revised fuzzy metric space. Then,

(a) If $lim_{n\to\infty} \zeta_n = \zeta \ and \ lim_{n\to\infty} \eta_n = \eta$, then

$$\mu(\zeta, \eta, \lambda - \epsilon) \geq lim_{n\to\infty} \sup \mu(\zeta_n, \eta_n, \lambda)$$

(b) If $lim_{n\to\infty} \zeta_n = \zeta \ and \ lim_{n\to\infty} \eta_n = \eta$, then

$$\mu(\zeta,\eta,\lambda+\epsilon) \leq \ lim_{n\to\infty} inf\ \mu(\zeta_n,\eta_n,\lambda)$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Proof for (a):

By the definition of Revised fuzzy metric space, conditions (iv)

$$\mu(\zeta_n,\eta_n,\lambda) \leq \ \mu\left(\zeta_n,\zeta,\frac{\in}{2}\right) \oplus(\zeta,\eta,\lambda - \in) \oplus \ \mu\left(\eta_n,\eta,\frac{\in}{2}\right)$$

$$lim_{n\to\infty} sup\ \mu(\zeta_n,\eta_n,\lambda) \ \leq \ 0 \oplus \ \mu(\zeta,\eta,\lambda - \in) \oplus 0$$

Hence, $\lim_{n\to\infty} sup\ \mu(\zeta_n,\eta_n,\lambda) \ \leq \ \mu(\zeta,\eta,\lambda - \in)$

Proof for (b):

By the definition of Revised fuzzy metric space, conditions (iv),

$$\mu(\zeta,\eta,\lambda+\in) \ \leq \mu\left(\zeta_n,\zeta,\frac{\in}{2}\right) \oplus \mu(\zeta_n,\eta_n,\in) \oplus \ \mu\left(\eta_n,\eta,\frac{\in}{2}\right)$$

$$\mu(\zeta,\eta,\lambda+\in) \leq \ \lim_{n\to\infty} inf\ \mu(\zeta_n,\eta_n,\in)$$

Corollary 3.3:

If $lim_{n\to\infty} \zeta_n \ = \ a\ and\ lim_{n\to\infty} \eta_n = \eta$

a) $\mu(\zeta,\eta,\lambda) \ \geq \ \lim_{n\to\infty} sup\ \mu(\zeta_n,\eta_n,\lambda)\ldots\ldots$ (3.3)

$\mu(\zeta,\eta,\lambda) \ \leq \ \lim_{n\to\infty} inf\ \mu(\zeta_n,\eta_n,\lambda)\ldots\ldots.$ (3.4)

for all $\lambda > 0$ and $0 < \in < \lambda$

Theorem 3.4: (Edelstein Contraction Theorem in Revised Fuzzy metric)

Let $(\Sigma,\mu,\oplus)$ be compact Revised Fuzzy metric space. Let $F : \Sigma \to \Sigma\ be$ a function satisfying

$$\mu(F\zeta,F\eta,.) \ < \ \mu(\zeta,\eta,.) \tag{3.5}$$

Then F has fixed point.

Proof:

Let $a \in \Sigma$ and $a_n = F^n \zeta\ (n \in N)$.

Assume $\zeta_n \neq \zeta_{n+1}$ for each n $(If\ not\ F\zeta_n = \zeta_n)$ consequently $a_n \neq a_{n+1}$ $(n \neq m)$, For otherwise we get

$$\mu(\zeta_n, \zeta_{n+1}, .) = \mu(\zeta_m, \zeta_{m+1}, .) < \mu(\zeta_{m-1}, \zeta_m, .) < \cdots < \mu(\zeta_n, \zeta_{n+1}, .)$$

where $m > n$, which is a contradiction.

Since Σ is compact set, $\{\zeta_n\}$ has convergent sub sequence $\{\zeta_{n_i}\}$.

Let $\eta = \lim_{i\to\infty} \zeta_{n_i}$, Also we assume that η such that $F\eta \in \{\zeta_{n_i} ; i \in N\}$.

According to the above assumption, we may now write

$$\mu(F\zeta_{n_i}, F\eta, .) < \mu(\zeta_{n_i}, \eta, .)$$

for all $i \in N$. Then by equation (3.3) we obtain

$$lim\ sup\ \mu(F\zeta_{n_i}, F\eta, \lambda) \leq lim\ \mu(\zeta_{n_i}, \eta, \lambda) = \mu(\eta, \eta, \lambda) = 0$$

for each $\lambda > 0$.

Hence,

$$lim\ F\zeta_{n_i} = F\eta .... \tag{3.6}$$

Similarly $lim\ F^2\zeta_{n_i} = F^2\eta ....$ (3.7)

(we recall that $lim\ F\zeta_{n_i} = F\eta$ for all $(i \in N)$, Now observe that,

$$\mu(\zeta_{n_i}, F\zeta_{n_i}, \lambda) \geq \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda)$$

$$\geq \Theta(F\zeta_{n_i}, F^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq \Theta(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda)$$

$$\geq \Theta(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0.$$

for all $\lambda > 0$. $\{\mu(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}$ and $\{\Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\}$ $(\lambda > 0)$ are convergent to a common limit point . So by equations (3.2) , (3.4) and (3.5) and we get,

$$\mu(\eta, z\eta, \lambda) \leq lim\ inf\ \mu(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \mu(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)$$

$$\leq lim\ \ sup\ \ \mu(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)$$

$$\leq \mu(F\eta, F^2\,\eta, \lambda)$$

for all $\lambda > 0$. Suppose $b \neq F\eta$, By equation (3.5)

$$\mu(\eta, F\eta, .) > \mu(F\eta, F^2\eta, .)$$

which is a contradiction , because the above function are right continuous ,non - increasing respectively.

Hence $\eta = F\eta$ is a fixed point.

To prove the uniqueness of the fixed point, let us consider $F(\zeta) = \omega$ for some $\zeta \in \Sigma$.
Then

$$0 \leq \mu(\zeta, \omega, \lambda) = \mu(F\eta, F\omega, \lambda) \leq \mu(\zeta, \omega, \frac{\lambda}{k}) = \mu(F\eta, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \mu(\zeta, \omega, \frac{\lambda}{k^n})$$

Now, we easily verify that $\{\frac{\lambda}{k^n}\}$ is an s-increasing sequence, then by assumption for a given $\epsilon \in (0,1)$, there exists $n_0 \in N$ such that

$$\mu(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon$$

Clearly,

$$\mu(\zeta, \omega, \lambda) = 0$$

Thus $\eta = \omega$. Hence proved.

CONCLUSION

The main purpose of this paper is to introduce a new class of Banach contraction and Edelstein Contraction **i**n revised fuzzy metric space and to present fixed point theorems.

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