BLOW-UP OF SOLUTIONS OF A LOGARITHMIC VISCOELASTIC PLATE EQUATION WITH DELAY TERM

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Abstract

In this paper, we consider a logarithmic viscoelastic plate equation with a delay term in a bounded domain. Under suitable conditions, we establish the blow-up of solutions in a finite time. Time delays often seem in thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine.

Keywords: *Blow-up, delay, logarithmic nonlinearity, viscoelastic plate equation.*

1. INTRODUCTION

In this paper, we consider the following logarithmic viscoelastic plate equatıon with delay term

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t g\,(t-s)\Delta^2 u(s)ds+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau) & \\ =ku|u|^{p-2}ln|u|, & x\in\Omega,\ t>0, \\ u(x,t)=\frac{\partial u(x,t)}{\partial \upsilon}=0, & x\in\partial\Omega,\ \ t\in[0,\infty), \\ u_t(x,t-\tau)=f_0(x,t-\tau), & in\ (0,\tau), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1}$$

where $\Omega\subset R^n$ $(n\geq 1)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. $p\geq 2$, k,μ_1 are positive constants, μ_2 is a real number, $\tau>0$ represents the time delay and the functions $u_0,\ u_1,\ f_0$ are the initial data to be specified later. υ is the unit outward normal vector to $\partial\Omega$. Without logarithmic source term $(ku|u|^{p-2}ln|u|)$, this model describes the deflection $u(x,t)$ of a viscoelastic beam (when $n=1$) or a viscoelastic plate (when $n=2$).

This type of logarithmic term $(u|u|^{p-2}ln|u|)$ arises naturally in nuclear physics, optics, geophysics, supersymmetric and inflation cosmology (see Bartkowski and Gorka (2008), Gorka (2009)). Problems about the mathematical behavior of solutions for PDEs with time delay effects have become interesting for many authors mainly because time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine. Moreover, it is well known that delay effects may destroy the stabilizing properties of a well-behaved system. In the literature, there are several examples that illustrate how time delays destabilize some internal or boundary control system (Hale (1993), Kafini and Messaoudi (2016)).

We consider the vibration of a viscoelastic beam, in one-dimensional space. The constitutive relationship between the stress N and strain u satisfies

$$N(x,t) = \alpha u_{xxx} - \int_0^t g(t-s)u_{xxx}(x,s)ds,$$

where the constant α represents the tension stiffness, and g is so-called relaxation function. We can get, if there exists the load $F(x,t,u,u_t)$ on the beam, the following model:

$$u_{tt} + \frac{\alpha}{\rho A}u_{xxxx} - \frac{\alpha}{\rho A}\int_0^t g(t-s)u_{xxxx}ds = \frac{F}{\rho A},$$

where ρ and A represent the density and the cross-sectional area of the beam, respectively.

We have the Euler-Bernoulli viscoelastic model (when $\frac{\alpha}{\rho A} = 1, F = 0$), in high-dimensional space, as follows:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0,$$

where Δ represents the Laplacian operator with respect to the spatial variables in R^n $(n \geq 2)$ and

$$\Delta^2 u = \Delta(\Delta u) = \sum_{j=1}^n \left(\sum_{i=1}^n u_{x_i x_i}\right)_{x_j x_j},$$

(Yang (2015).

In 1986, Datko et al. indicated that delay is a source of instability. Nicaise and Pignotti (2006) considered the following wave equation with a linear damping and delay term

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0. \qquad (2)$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. In the absence of delay, Zuazua (1990) looked into exponentially stability for the equation (2).

Kirane and Said-Houari (2011), studied the viscoelastic wave equation as follows:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \tag{3}$$

with suitable initial-boundary value conditions and proved the well-posedness and the energy decay of solutions, under the restriction $0 < \mu_2 \leq \mu_1$. In the presence of the logarithmic source term $(u|u|^{p-2}ln|u|^k)$, Pişkin and Yüksekkaya (2021a) established the local existence and proved the blow up of solutions in a finite time of the equation (3).

Cavalcanti et al. (2001), studied the model as follows:

$$u_{tt} + \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + a(t)u_t = 0, \tag{4}$$

in $\Omega \times (0,\infty)$, where $a(t)$ is a nonlocal nonlinearity type function. They established the exponential decay result when $\gamma = 0$, of the energy in general domains of (4). Rivera et al. (1996), coupled (4) with a dynamic boundary condition and indicated that the sum of the first and second energies decay polynomially and exponentially, according as the relaxation function g decays polynomially or exponentially. Also, for more results on (4), see also Lagnese (1989).

Mukiawa (2020), considered the viscoelastic plate equation as follows:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \tag{5}$$

with a constant time delay and partially hinged boundary condition. The author proved a general decay result of the equation (5). Yang (2015), proved the global solution of the equation (5) under suitable assumptions.

Mustafa and Kafini (2017), studied the infinite memory-type plate equation in the presence of constant time delay as follows:

$$u_{tt} + \Delta^2 u - \int_0^\infty g(s)\Delta^2 u(t-s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = u|u|^\gamma. \tag{6}$$

The authors proved an explicit and general decay result for the energy, under the condition that $|\mu_2| \leq \mu_1$, without restrictive assumptions on the behavior of the relaxation function g at infinity of the equation (6).

Kafini and Messaoudi (2020), considered the problem:

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = u|u|^{p-2}ln|u|^k. \tag{7}$$

They studied the local existence and the blow up result of the equation (7). In recent years, some other authors investigate hyperbolic and parabolic type equations (see Antontsev et al. (2021), Al-Gharabli et al. (2020), Ferreira et al.(2021), Mezouar et al. (2020), Pişkin and Polat (2013), Pişkin and Yüksekkaya (2018a), (2018b), (2018c), Pişkin and Yüksekkaya (2020a), (2020b), Pişkin and Cömert (2020), Pişkin and Çalışır (2020), Pişkin and Yüksekkaya (2021a), (2021b), (2021c), (2021d), (2021e), (2021f), (2021g), Yüksekkaya et al. (2021a), (2021b), Yüksekkaya and Pişkin (2021c)).

To our best knowledge, there is no research on the viscoelastic plate equation with delay term and logarithmic source term. The aim of the present paper is to establish the sufficient conditions for the blow up to the nonlinear logarithmic viscoelastic plate equation with delay term. We extend the classical logarithmic wave equation to logarithmic viscoelastic plate equation.

The paper is organized as follows: In section 2, we give some materials that will be used later. In section 3, we state and prove our main result.

2. PRELIMINARIES

In this part, we prepare some materials for the proof of our result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Now, we give some assumptions used in this paper.

Assume that $g:[0,\infty) \to [0,\infty)$ is a nonincreasing and differentiable function satisfying

$$1 - \int_0^\infty g(s)\,ds = l > 0. \tag{8}$$

We make the following extra assumptions on g

$$g'(s) \leq 0 \; and \; \int_0^t g(s)\,ds < \frac{\frac{p(1-a)-2}{2} - \frac{C|\mu_2|^2}{4\kappa C_0}}{1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}}. \tag{9}$$

Let $B_p > 0$ be the constant satisfying (Adams (2003), Pişkin (2017))

$$\|\nabla \upsilon\|_p \leq B_p \|\Delta \upsilon\|_p, \; for \; \upsilon \in H_0^2(\Omega). \tag{10}$$

It holds

$$\int_0^t g(t-s)\big(\Delta u(s), \Delta u_t(t)\big)\,ds = -\frac{1}{2} g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' o \Delta u)(t)$$

$$-\frac{1}{2}\frac{d}{dt}\left[(g o \Delta u)(t) - \left(\int_0^t g(s)\,ds\right)\|\Delta u(t)\|^2\right], \tag{11}$$

where

$$(go\Delta u)(t)=\int_0^t g(t-s)\,\|\Delta u(t)-\Delta u(s)\|^2 ds.$$

As in Nicaise and Pignotti (2008), we introduce the new variable

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\qquad x\in\Omega,\qquad \rho\in(0,1),\qquad t>0.$$

Thus, we have

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\qquad x\in\Omega,\qquad \rho\in(0,1),\qquad t>0.$$

Therefore, problem (1) takes the form:

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t g\,(t-s)\Delta^2 u(s)ds+\mu_1 u_t+\mu_2 z(x,1,t) & \\ =ku|u|^{p-2}ln|u|, & in\ \Omega\times(0,\infty),\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & in\ \Omega\times(0,1)\times(0,\infty)\\ z(x,\rho,0)=f_0(x,-\rho\tau), & in\ \Omega\times(0,1),\\ u(x,t)=\frac{\partial u(x,t)}{\partial \upsilon}=0, & x\in\partial\Omega,\ \ t\in[0,\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & in\ \Omega, \end{cases} \tag{12}$$

The energy functional associated with problem (12) is

$$E(t)=\tfrac{1}{2}\|u_t\|^2+\tfrac{1}{2}(go\Delta u)(t)+\tfrac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\Delta u\|^2$$

$$+\tfrac{k}{p^2}\|u\|_p^p-\tfrac{k}{p}\int_\Omega |u|^p\,ln|u|dx+\tfrac{\xi}{2}\int_\Omega\int_0^1|z(x,\rho,t)|^2\,d\rho dx, \tag{13}$$

where

$$\tau|\mu_2|<\xi<\tau(2\mu_1-|\mu_2|),\qquad |\mu_2|<\mu_1. \tag{14}$$

We also set

$$H(t)=-E(t)=-\frac{1}{2}\|u_t\|^2-\frac{1}{2}(go\Delta u)(t)-\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\Delta u\|^2$$

$$-\frac{k}{p^2}\|u\|_p^p+\frac{k}{p}\int_\Omega |u|^p\,ln|u|dx-\frac{\xi}{2}\int_\Omega\int_0^1|z(x,\rho,t)|^2\,d\rho dx,$$

to prove our main result.

The following lemma gives that the associated energy of the problem under the condition $\mu_1>|\mu_2|$ is decrasing.

Lemma 1. Let u be the solution of (12). Then, for some $C_0 \geq 0$,

$$E'(t) \leq -C_0 \int_{\Omega} (|u_t|^2 + |z(x,1,t)|^2)\, dx + \frac{1}{2}(g' o\Delta u)(t) - \frac{1}{2} g(t)\|\Delta u(t)\|^2$$

$$\leq -C_0 \int_{\Omega} (|u_t|^2 + |z(x,1,t)|^2)\, dx \leq 0. \qquad (15)$$

Proof. Multiplying the equation (12) by u_t and integrating over $\mathbf{\Omega}$, and use integration by parts, we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|^2 + \frac{1}{2}(go\Delta u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\Delta u\|^2\right.$$

$$\left. + \frac{k}{p^2}\|u\|_p^p - \frac{k}{p}\int_{\Omega} |u|^p \, ln|u| dx\right\}$$

$$= -\frac{1}{2} g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' o\Delta u)(t)$$

$$-\mu_1 \int_{\Omega} |u_t(t)|^2 dx - \mu_2 \int_{\Omega} u_t z(x,1,t) dx. \qquad (16)$$

We multiply the second equation in (12) by $(\xi/\tau)z$ and integrate over $\Omega \times (0,1)$, $\xi > 0$, we have

$$\frac{\xi}{2}\frac{d}{dt}\int_{\Omega}\int_0^1 z^2(x,\rho,t)\, d\rho dx + \frac{\xi}{\tau}\int_{\Omega}\int_0^1 z(x,\rho,t) z_\rho(x,\rho,t) d\rho dx = 0. \qquad (17)$$

Noting that

$$-\frac{\xi}{\tau}\int_{\Omega}\int_0^1 z(x,\rho,t)\, z_\rho(x,\rho,t)\, d\rho dx$$

$$= -\frac{\xi}{2\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial \rho} z^2(x,\rho,t)\, d\rho dx$$

$$= \frac{\xi}{2\tau}\int_{\Omega} \left(z^2(x,0,t) - z^2(x,1,t)\right) dx$$

$$= \frac{\xi}{2\tau}\left(\int_{\Omega} |u_t|^2 dx - \int_{\Omega} z^2(x,1,t)\, dx\right). \qquad (18)$$

Combining (16) and (17) and taking into consideration (18), we have

$$E'(t) = -\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_{\Omega} |u_t(x,t)|^2 dx - \frac{\xi}{2\tau}\int_{\Omega} |z^2(x,1,t)|\, dx$$

$-\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' o \Delta u)(t) - \mu_2 \int_\Omega u_t\, z(x,1,t)dx,$ (19)

for $t \in (0,T)$.

Utilizing Young's inequality, we estimate

$$-\mu_2 \int_\Omega z(x,1,t)u_t(x,t)\,dx \leq \frac{|\mu_2|}{2}\int_\Omega (|u_t(x,t)|^2 + |z(x,1,t)|^2)\,dx.$$

Hence, from (19), we have

$$E'(t) \leq -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |u_t(x,t)|^2\,dx$$

$$-\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |z(x,1,t)|^2\,dx$$

$+\frac{1}{2}(g' o\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2.$ (20)

By using (14), for some $C_0 > 0$, we obtain

$$E'(t) \leq -C_0 \int_\Omega (|u_t(x,t)|^2 + |z(x,1,t)|^2)\,dx$$

$$+\frac{1}{2}(g' o\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2$$

$$\leq -C_0 \int_\Omega (|u_t(x,t)|^2 + |z(x,1,t)|^2)\,dx \leq 0.$$

To get our main result, we have the following lemmas.

Lemma 2. There exists a positive constant $C > 0$ depending on Ω only such that

$$\left(k\int_\Omega |u|^p\, ln|u| dx\right)^{s/p} \leq C\left[k\int_\Omega |u|^p\, ln|u|dx + \|\Delta u\|^2\right],$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega |u|^p\, ln|u|^k dx \geq 0$.

Proof. In Kafini and Messaoudi (2020), by Lemma 3.2 we know that

$$\left(\int_\Omega |u|^p\, ln|u|^k dx\right)^{s/p} \leq C\left[\int_\Omega |u|^p\, ln|u|^k dx + \|\nabla u\|^2\right],$$

is satisfied, by using the Sobolev Embedding Theorem we get the result.

Similar to Kafini and Messaoudi (2020) and by using the Sobolev Embedding Theorem, we have the following lemmas.

Lemma 3. There exists a positive constant $C > 0$ depending on Ω only such that

$$\|u\|^2 \leq C\left[\left(k\int_\Omega |u|^p ln|u|\, dx\right)^{\frac{2}{p}} + \|\Delta u\|_2^{\frac{4}{p}}\right], \qquad (21)$$

provided that $\int_\Omega |u|^p \, ln|u|^k dx \geq 0$.

Lemma 4. There exists a positive constant $C > 0$ depending on Ω only such that

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\Delta u\|^2\right], \qquad (22)$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p$.

3. BLOW UP RESULTS

In this part, we prove the blow up of solutions in a finite time for the problem (12).

Theorem 5. Assume that the condition (14) hold. Let

$$\begin{cases} 2 \leq p, & if\ n = 1,2,3,4, \\ 2 \leq p \leq \dfrac{2(n-2)}{n-4}, & if\ n \geq 5, \end{cases}$$

and

$$E(0) < 0. \qquad (23)$$

Then the solution of (12) blows up in finite time.

Proof. Reminding (15), we get

$$E(t) \leq E(0) < 0.$$

Hence,

$$H'(t) = -E'(t) = C_0 \int_0^1 \left(|u_t|^2 + z^2(x,1,t)\right)dx$$

$$-\frac{1}{2}(g' o \Delta u)(t) + \frac{1}{2} g(t)\|\Delta u\|^2$$

$$\geq C_0 \int_0^1 z^2(x,1,t)dx \geq 0, \tag{24}$$

and

$$0 < H(0) \leq H(t) \leq \frac{k}{p}\int_\Omega |u|^p ln|u|\, dx. \tag{25}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega uu_t\, dx + \frac{\mu_1 \varepsilon}{2}\int_\Omega u^2\, dx, \ \ t \geq 0,$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{26}$$

Differentiating $L(t)$ with respect to t, we get

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon\int_\Omega uu_{tt}\, dx + \mu_1\varepsilon\int_\Omega uu_t\, dx$$

$$= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon k\int_\Omega |u|^p ln|u|\, dx - \varepsilon\|\Delta u\|^2$$

$$+\varepsilon\left(\int_\Omega \int_0^t g(t-s)\big(\Delta u(s)\Delta u(t)\big)dsdx\right) - \varepsilon\mu_2\int_\Omega uz(x,1,t)\, dx. \tag{27}$$

Thanks to Young's inequality, we have

$$-\varepsilon\mu_2\int_\Omega uz(x,1,t)\, dx \leq \varepsilon|\mu_2|\left(\delta\int_\Omega u^2\, dx + \frac{1}{4\delta}\int_\Omega z^2\,(x,1,t)dx\right), \forall\delta > 0, \tag{28}$$

and $\forall \eta > 0$,

$$\int_0^t g(t-s)\,\big(\Delta u(s),\Delta u(t)\big)ds = \int_0^t g(t-s)\,\big(\Delta u(s)-\Delta u(t),\Delta u(t)\big)ds$$

$$+\int_0^t g(t-s)\,\|\Delta u(t)\|^2 ds$$

$$\geq \left(1-\frac{1}{4\eta}\right)\int_0^t g(s)ds\,\|\Delta u(t)\|^2 - \eta(go\Delta u)(t). \tag{29}$$

Inserting (28) and (29) into (27), we have

$$L'(t) \geq \left[(1-\alpha)H^{-\alpha}(t) - \frac{\varepsilon|\mu_2|}{4\delta C_0}\right]H'(t) + \varepsilon\|u_t\|^2 + \varepsilon k\int_\Omega |u|^p ln|u|\, dx$$

$$+\varepsilon\left(\left(1-\frac{1}{4\eta}\right)\int_0^t g(s)ds-1\right)\|\Delta u\|^2$$

$$-\varepsilon\eta(go\Delta u)(t)-\varepsilon\delta|\mu_2|\|u\|^2. \quad (30)$$

By taking δ so that $|\mu_2|/4\delta C_0=\kappa H^{-\alpha}(t)$, for large κ to be specified later and substituting in (30), we obtain

$$L'(t)\geq[(1-\alpha)-\varepsilon\kappa]H^{-\alpha}(t)H'(t)+\varepsilon\|u_t\|^2$$

$$+\varepsilon\left(\left(1-\frac{1}{4\eta}\right)\int_0^t g(s)ds-1\right)\|\Delta u\|^2$$

$$-\varepsilon\eta(go\Delta u)(t)-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^{\alpha}(t)\|u\|^2$$

$$+\varepsilon k\int_\Omega |u|^p ln|u|\,dx.$$

For $0<\alpha<1$, we get

$$L'(t)\geq[(1-\alpha)-\varepsilon\kappa]H^{-\alpha}(t)H'(t)$$

$$+\big(\varepsilon-\varepsilon(1-\alpha)\big)k\int_\Omega |u|^p ln|u|\,dx+\varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2$$

$$+\varepsilon\left(\left(1-\frac{1}{4\eta}-\frac{p(1-a)}{2}\right)\int_0^t g(s)ds+\frac{p(1-a)-2}{2}\right)\|\Delta u\|^2$$

$$-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^{\alpha}(t)\|u\|^2+\varepsilon p(1-a)H(t)$$

$$+\frac{\varepsilon k(1-a)}{p}\|u\|_p^p+\varepsilon\left(-\eta+\frac{p(1-a)}{2}\right)(go\Delta u)(t)$$

$$+\frac{\varepsilon(1-a)p\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \quad (31)$$

Thanks to (21) and (25), we obtain

$$H^{\alpha}(t)\|u\|_2^2\leq\left(\int_\Omega |u|^p ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2$$

$$\leq \left[(|u|^p ln|u|^k dx)^{\alpha+\frac{2}{p}} + (|u|^p ln|u|^k dx)^{\alpha}\|\Delta u\|_2^{\frac{4}{p}}\right].$$

By using Young's inequality, we have

$$H^{\alpha}(t)\|u\|_2^2 \leq \left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\alpha} \|u\|_2^2$$

$$\leq \left[\left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\frac{(p\alpha+2)}{p}} + \frac{2}{p}\|\Delta u\|^2 + \frac{p-2}{p}\left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\frac{\alpha p}{(p-2)}}\right].$$

Therefore, we obtain

$$H^{\alpha}(t)\|u\|_2^2 \leq \left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\alpha} \|u\|_2^2$$

$$\leq C\left[\left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\frac{(p\alpha+2)}{p}} + \|\Delta u\|^2 + \left(\int_{\Omega} |u|^p ln|u|^k \, dx\right)^{\frac{\alpha p}{(p-2)}}\right],$$

where C=max$\left\{\frac{2}{p}, \frac{p-2}{p}\right\}$.

From (26), we have

$$2 < \alpha p + 2 \leq p \ and \ 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Hence, Lemma 2 provides

$$H^{\alpha}(t)\|u\|_2^2 \leq C\left(k\int_{\Omega} |u|^p ln|u|^k \, dx + \|\Delta u\|_2^2\right). \tag{32}$$

Combining (31) and (32), we obtain

$$L'(t) \geq [(1-\alpha) - \varepsilon\kappa]H^{-\alpha}(t)H'(t) + \varepsilon\left(a - \frac{C|\mu_2|^2}{4\kappa C_0}\right)\int_{\Omega} |u|^p ln|u|^k \, dx$$

$$+\varepsilon\left(\left(1 - \frac{1}{4\eta} - \frac{p(1-a)}{2}\right)\int_0^t g(s)ds + \frac{p(1-a)-2}{2} - \frac{C|\mu_2|^2}{4\kappa C_0}\right)\|\Delta u\|^2$$

$$+\varepsilon\left(-\eta + \frac{p(1-a)}{2}\right)(go\Delta u)(t) + \varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2 + \varepsilon\frac{k(1-a)}{p}\|u\|_p^p$$

$$+\varepsilon p(1-a)H(t)+\varepsilon\frac{(1-a)p\xi}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx. \tag{33}$$

Now, we choose $a>0$ so small that

$$\frac{p(1-a)+2}{2}>0,$$

And κ so large that

$$\left(1-\frac{1}{4\eta}-\frac{p(1-a)}{2}\right)\int_0^t g(s)ds+\frac{p(1-a)-2}{2}-\frac{C|\mu_2|^2}{4\kappa C_0}>0,$$

and

$$\begin{cases} a-\dfrac{C|\mu_2|^2}{4\kappa C_0}>0, \\ -\eta+\dfrac{p(1-a)}{2}>0. \end{cases}$$

Once κ and a are fixed, we choose ε so small so that

$$(1-\alpha)-\varepsilon\kappa>0,$$

$$H(0)+\varepsilon\int_{\Omega} u_0u_1\,dx>0.$$

Thus, for some $\lambda>0$, the estimate (33) becomes

$$L'(t)\geq\lambda\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+(go\Delta u)(t)+\|u\|_p^p\right]$$

$$+\lambda\left[\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx+k\int_{\Omega}|u|^p ln|u|^k\,dx\right] \tag{34}$$

and

$$L(t)\geq L(0)>0,\ t\geq 0. \tag{35}$$

Next, using Hölder's inequality and the embedding $\|u\|_2\leq C\|u\|_p$, we obtain

$$\int_{\Omega} uu_t\,dx\leq\|u\|_2\|u_t\|_2\leq C\|u\|_p\|u_t\|_2,$$

and exploiting Young's inequality, we have

$$\left|\int_{\Omega} uu_t\,dx\right|^{1/(1-\alpha)}\leq C\left(\|u\|_p^{\frac{\mu}{(1-\alpha)}}+\|u_t\|_2^{\frac{\theta}{(1-\alpha)}}\right),\ for\frac{1}{\mu}+\frac{1}{\theta}=1. \tag{36}$$

By using Lemma 4, we take $\theta = 2(1-\alpha)$ which satisfies $\frac{\mu}{(1-\alpha)} = \frac{2}{(1-2\alpha)} \leq p$. Therefore, for $s = \frac{2}{(1-2\alpha)}$, estimate (36) yields

$$\left|\int_\Omega u u_t \, dx\right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^s + \|u_t\|_2^2\right).$$

Thus, Lemma 4 gives

$$\left|\int_\Omega u u_t \, dx\right|^{1/(1-\alpha)} \leq C\left[\|\Delta u\|^2 + \|u_t\|_2^2 + \|u\|_p^p\right]. \tag{37}$$

Hence,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t \, dx + \frac{\mu_1 \varepsilon}{2} \int_\Omega u^2 \, dx\right)^{1/(1-\alpha)} \\ &\leq C\left[H(t) + \left|\int_\Omega u u_t \, dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\ &\leq C\left[H(t) + \left|\int_\Omega u u_t \, dx\right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)}\right] \end{aligned}$$

$$\leq \left[H(t) + \|\Delta u\|^2 + \|u_t\|^2 + \|u\|_p^p\right], \; t \geq 0. \tag{38}$$

Combining (34) and (38), we get

$$L'(t) \geq \Lambda L^{\frac{1}{(1-\alpha)}}(t), \; t \geq 0, \tag{39}$$

where Λ is a positive constant depending only on λ and C.

An integration of (39) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\frac{\alpha}{(1-\alpha)}}(0) - \frac{\Lambda \alpha t}{(1-\alpha)}}.$$

Thus, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda \alpha L^{\frac{\alpha}{(1-\alpha)}}(0)}.$$

As a result, the solution of problem (12) blows up in finite time T^* and $T^* \leq \frac{1-\alpha}{\Lambda \alpha L^{\frac{\alpha}{(1-\alpha)}}(0)}$.

CONCLUSIONS

Recently, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no blow-up of solutions for the logarithmic viscoelastic plate equation with delay term. Under suitable conditions, we have been proved the blow-up of solutions.

ACKNOWLEDGEMENT The authors are grateful to DUBAP (ZGEF.20.009) for research funds.

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