RECOVERING THE INITIAL CONDITION OF PARABOLIC EQUATIONS FROM LATERAL CAUCHY DATA AS A GENERALIZED PROBLEM OF MOMENTS

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Abstract

It will be shown that to recover the initial condition and finding the solution of a parabolic equation with Cauchy conditions can be solved in two steps writing the parabolic equation as an integral equation, which can be solved numerically applying the techniques of inverse generalized moments problem. In a first step, the initial condition is found in approximate form, and in a second step we numerically approximate the solution of the parabolic equation using the initial condition found.
The method is illustrated with examples.

Keywords:*parabolic equation, integral equations, generalized moment problem, inverse problem*.

INTRODUCTION

We want to find $w(x,t)$ and $\varphi(x)$ such that

$$w_t - w_{xx} = R(x,t) \text{ and } w(x,0) = \varphi(x)$$

with $R(x,t)$ known about a D domain where $D = \{(x,t); a_1 < x < b_1, t > 0\}$.
The underlying space is $L^2(D)$ under the conditions

$$w(a_1,t) = s_1(t) \qquad w(b_1,t) = s_2(t) \quad t \geq 0$$

$$w_x(a_1,t) = k_1(t) \qquad w_x(b_1,t) = k_2(t) \quad t \geq 0$$

The problem has been studied and solved, to give an example, with the quasi-reversibility method [1]. Quoting verbatim the introduction of [1]: "This problem has many real-world applications; for e.g., determine the spatially distributed temperature inside a solid from the boundary measurement of the heat and heat flux in the time domain [2]; identify the pollution on the surface of the rivers or lakes [3]; effectively monitor the heat conductive processes in steel industries, glass and polymer forming and nuclear power station [4]. Due to its realistic applications, this problem has been studied intensively".

Many researchers studied the problem of determining the initial value in a problem with partial differential equations. In [5] is studied the inverse problem of identifying the initial value of a two-dimensional degenerate parabolic equation. In [6] the problem of recover the initial temperature of a body from discrete temperature measurements made at later times is studied.
In [7] it is proposed a new numerical method for the solution of the problem of the reconstruction of the initial condition of a quasilinear parabolic equation from the measurements of both Dirichlet and Neumann data on the boundary of a bounded domain. In [8] approaches to the numerical recovering of the initial condition in the inverse problem a for a nonlinear singularly perturbed reaction–diffusion–advection equation are considered. In [9] is studied the initial boundary value problem for a class of higher-order n-dimensional nonlinear pseudo-parabolic equations which do not have positive energy and come from the soil mechanics, the heat conduction, and the nonlinear optics.
The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. The interest is not to compare with the existing methods, but to present a different method to my novel criteria, and theone that I have already applied in other cases of partial differential equations under other conditions, for example the Parabolic equation under Cauchy conditions or Dirichlet or Neumann. It turns out that a change in conditions implies a different approach. This is a significant change in the problem statement for its resolution.

The generalized moments problem [10,11,12], is to find a function $f(x)$ about a domain $\Omega \subset R^d$ that satisfies the sequence of equations

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i \epsilon N ---------- \quad (1)$$

where N is the set of the natural numbers, $(g_i(x))$ is a given sequence of functions in $L^2(\Omega)$ linearly independent known and the succession of real numbers $\{\mu_i\}_{i \epsilon N}$ are known data.

The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [10,11,12]. There are several methods to build regularized solutions. One of them is the truncated expansion method [10].

This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i = 1,2,\dots,n. ------- \quad (2)$$

where it is considered as approximate solution of to $p_n(x) = \sum_{i=0}^{n} \lambda_i \phi_i(x)$, and the functions $\{\phi_i(x)\}_{i=1,..,n}$ result of orthonormalize $\langle g_1, g_2, \dots, g_n \rangle$ being λ_i the coefficients based on the data μ_i. In the subspace generated by $\langle g_1, g_2, \dots, g_n \rangle$ the solution is stable. If $n \epsilon N$ is chosen in an appropriate way then the solution of (2) it approaches the solution of the problem (1).

In the case where the data $\mu_i$are inaccurate the convergence theorems should be applied and error estimates for the regularized solution (pages 19 - 30 of [10]).

ARTICLE ORGANIZATION

To find $w(x,t)$ and $\varphi(x)$ such that $w_t - w_{xx} = R(x,t)$ and $w(x,0) = \varphi(x)$

with $R(x,t)$ known about a D domain where $D = \{(x,t); a_1 < x < b_1, t > 0\}$ under the conditions

$$w(a_1,t) = s_1(t) \qquad w(b_1,t) = s_2(t) \quad t \geq 0$$

$$w_x(a_1,t) = k_1(t) \qquad w_x(b_1,t) = k_2(t) \quad t \geq 0$$

we will do it in two steps. The next section describes the first step. The section that follows explains the second step. Then it is explained how the generalized moment problem is solved with the truncated expansion method. Finally the numerical example and the conclusions.

APPROXIMATION OF THE INITIAL CONDITION φ(x) - FIRST STEP

We consider

$$w_t - w_{xx} = R(x,t) \;-----\quad (2)$$

the conditions are:

$$w(a_1,t) = s_1(t) \qquad w(b_1,t) = s_2(t) \quad t \geq 0$$

$$w_x(a_1,t) = k_1(t) \qquad w_x(b_1,t) = k_2(t) \quad t \geq 0$$

We take as an auxiliary function

$$u(m,r,x,t) = e^{-\frac{m}{b1}x}e^{-rt}$$

Note that $u_{xx} = \left(\frac{m}{b_1}\right)^2 u$ and $u_t = -ru$.

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w)$$

Since $div(F^*) = -R(x,t)$ we have to:

$$\iint_D u\, div(F^*) dA = \iint_D u\,\big(-R(x,t)\big) dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so

$$\iint_D u\, div(F^*)dA = \iint_D div(uF^*)dA - \iint_D F^*.\nabla u dA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_D F^*.\nabla u dA = \iint_D (F_1 u_x + F_2 u_t)dA$$

Integrating by parts with respect to x:

$$\iint_D F_1\, u_x dA = \int_0^\infty \int_{a_1}^{b_1} F_1\, u_x dxdt =$$

$$= \int_0^\infty \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt - \iint_D wu_{xx}\, dA =$$

$$\int_0^\infty \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt - \iint_D w\left(\frac{m}{b_1}\right)^2 udA$$

Besides

$$\iint_D F_2\, u_t dA = \int_0^\infty \int_{a_1}^{b_1} F_2\, u_t dxdt = \int_0^\infty \int_{a_1}^{b_1} (-w)(-ru)dxdt$$

then

$$\iint_D F^*.\nabla u dA = \int_0^\infty \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt -$$

$$-\iint_D w\, u\left(\left(\frac{m}{b_1}\right)^2 - r\right)dA = A(m,r) - \iint_D w\, u\left(\left(\frac{m}{b_1}\right)^2 - r\right)dA$$

where

$$A(m,r) = \int_0^\infty \big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt$$

On the other hand,

$$\int_C (uF^*) . nds =$$

$$= \int_{a_1}^{b_1} u(m,r,x,0)w(x,0)dx + \int_0^{\infty} u(m,r,b_1,t)w_x(b_1,t)dt - \int_0^{\infty} u(m,r,a_1,t)w_x(a_1,t)dt$$

$$= G(m,r)$$

$$\therefore \iint u\left(-R(x,t)\right)dA = G(m,r) - A(m,r) + \iint_D w\,u\left(\left(\frac{m}{b_1}\right)^2 - r\right)dA$$

So if we do $r = \left(\frac{m}{b_1}\right)^2$:

$$\int_{a_1}^{b_1} u\left(m, \left(\frac{m}{b_1}\right)^2, x, 0\right) w(x,0)dx = A\left(m, \left(\frac{m}{b_1}\right)^2\right) -$$

$$- \int_0^{\infty} u\left(m, \left(\frac{m}{b_1}\right)^2, b_1, t\right) w_x(b_1,t)dt - \int_0^{\infty} u\left(m, \left(\frac{m}{b_1}\right)^2, a_1, t\right) w_x(a_1,t)dt -$$

$$- \iint u\left(m, \left(\frac{m}{b_1}\right)^2, x, t\right) R(x,t)dA = \phi(m)$$

To solve this integral equation we give integer values to m, $m = 0,1,2, \ldots n$

Then

$$\int_{a_1}^{b_1} \varphi(x)H_m(x)dx = \phi(m) = \mu_m - - - - - - - - (3)$$

We interpret (3) as a generalized moments problem.

$p_{1n}(x)$ is the numerical approximation found with the truncated expansion method for $\varphi(x) = w(x,0)$, with $H_m(x) = u\left(m, \left(\frac{m}{b_1}\right)^2, x, 0\right) = e^{-\frac{m}{b_1}x}$ $m = 0,1,2, \ldots n$ where n is conveniently chosen.

In section 4 the truncated expansion method will be explained in detail and a theorem will be given that explains what is the accuracy of the approximation found by this method.

APPROACH TO $w(x,t)$ - SECOND STEP

To find an approximation of $w(x,t)$ a similar approach to the previous one is made where $w(x,0)$ is replaced by $p_{1n}(x)$ and we do not consider $r = \left(\frac{m}{b_1}\right)^2$.

We take the auxiliary function $u(m,r,x,t) = e^{-\frac{m}{b_1}x}e^{-(r+1)t}$.

Note that $u_{xx} = \left(\frac{m}{b_1}\right)^2 u$ and $u_t = -(r+1)u$.

We define the vector field $F^* = \left(F_1(w), F_2(w)\right) = (w_x, -w)$

Since $div(F^*) = -R(x,t)$ we have to:

$$\iint_D u\, div(F^*)dA = \iint_D u\left(-R(x,t)\right)dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so:

$$\iint_D u\, div(F^*)dA = \iint_D div(uF^*)dA - \iint_D F^*.\nabla u dA$$

Thus

$$\therefore \iint u\left(-R(x,t)\right)dA = G(m,r) - A(m,r) + \iint_D w\,u\left(\left(\frac{m}{b_1}\right)^2 - (r+1)\right)dA$$

Then

$$\iint_D w\,u\left(\left(\frac{m}{b_1}\right)^2 - (r+1)\right)dA = -G(m,r) + A(m,r) + \iint_D u\,R(x,t)dA$$

where now

$$G(m,r) = \int_{a_1}^{b_1} u(m,r,x,0)p_{1n}(x)dx + \int_0^{\infty} u(m,r,b_1,t)w_x(b_1,t)dt -$$

$$-\int_0^{\infty} u(m,r,a_1,t)w_x(a_1,t)dt$$

Equivalently

$$\iint_D w(x,t)H_{mr}(x,t)dA = \frac{-G(m,r) + A(m,r) + \iint_D u\,R(x,t)dA}{\left(\left(\frac{m}{b_1}\right)^2 - (r+1)\right)} = \phi(m,r)$$
$$= \mu_{mr} ----(4)$$

where

$$H_{m,r}(x) = u(m,r,x,t)$$

We can consider (4) as a two-dimensional generalized moment problem if we discretize giving m and r non-negative integer values $m = 0,1,2, \dots n_1$; $r = 0,2, \dots, n_2$, where n_1 and n_2 are conveniently chosen.

An approximation $p_{2n}(x,t)$ is found by the truncated expansion method for $w(x,t)$ where $n = n_1 \times n_2$.

It could happen that when discretizing the function ϕ we find that it is discontinuous in some integer values. In that case we can apply the following:

If we are in the case of a two-dimensional problem, we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)} \qquad i = 0,1, \dots, n_1 \quad j = 0,1,2, \dots, n_2$$

and we multiply both members of (4) by $\psi_{ij}(m,r)$ and we integrate with respect to m and r.

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_0^{\infty} w(x,t)K_{ij}(x,t) = \mu_{ij} -------- (5)$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_0^{\infty} \phi(m,r)\psi_{ij}(m,r)dmdr$$

$$K_{ij}(x,t) = \int_{a_1}^{b_1}\int_0^{\infty} u(m,r,x,t)\psi_{ij}(m,r)dmdr$$

We apply the truncated expansion method and find a numerical approximation for $w(x,t)$.

Analogously if we are in the one-dimensional case. The base would be

$$\psi_i(m) = m^i \qquad i = 0,1, \dots, n$$

The following section details the truncated expansion method for the two-dimensional case.

The one-dimensional case is analogous.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [12] and generalized in [13] and [14] to find an approximation $p_n(x,t)$ for the corresponding finite problem with $i = 0,1,2,\dots,n$ where n is the number of moments μ_i. We consider the basis $\phi_i(x,t)$ $i = 0,1,2,\dots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t)$ $i = 0,1,2,\dots,n$.

We approximate the solution $w(x,t)$ with [12] and generalized in [13] y [14]:

$$p_n(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t)$$

where

$$\lambda_i = \sum_{j=0}^{i} C_{ij}\, \mu_j \quad i = 0,1,2,\dots,n$$

And the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) | \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) . \|\phi_i(x,t)\|^{-1} \quad 1 < i \le n \, ; 1 \le j < i \, .$$

The terms of the diagonal are $\|\phi_i(x,t)\|^{-1}$ $i = 0,1,\dots,n$.

The proof of the following theorem is in [14,15].

In [15] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [16] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

Theorem

We considerer $b_2 = \infty$.Sea $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2((a_1,b_1) \times (a_2,b_2))$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n} \left| \iint_E H_i\,(x,t) f(x,t) dt dx - \mu_i \right|^2 \le \varepsilon^2 \quad .$$

$$\iint_E \left(x\, {f_x}^2 + t\, {f_t}^2\right) Exp[x+t]dtdx \leq M^2 -------- (5).$$

And it must be fulfilled that

$$t^i f(x,t) \to 0 \quad \text{si} \quad t \to \infty \quad \text{para todo} \quad i \in N$$

then

$$\int_{a_1}^{b_1} \int_{a_2}^{\infty} |f(x,t)|^2\, dtdx \leq min_i \left\{ \|C^T C\| \varepsilon^2 + \frac{1}{8(n+1)^2} M^2; i = 0,1,\ldots,n \right\}$$

where C it is a triangular matrix with elements $C_{ij} \quad (1 < i \leq n;\ 1 \leq j < i)$

and

$$\int_{a_1}^{b_1} \int_{a_2}^{\infty} |f(x,t) - p_n(x,t)|^2\, dtdx \leq \|C^T C\| \varepsilon^2 + \frac{1}{8(n+1)^2} M^2.$$

If b_2 it is not infinite then (5) change by

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} ((b_1 - a_1)^2 f_x^2 + (b_2 - a_2)^2 f_t^2) dx\, dt \leq M^2$$

NUMERICAL EXAMPLES

Example 1

We consider the equation

$$w_t - w_{xx} = 0 \qquad \text{in} \quad 0 < x < \pi; \quad t > 0$$

whose solution is: $w(x,t) = sin(x)e^{-t} \qquad \varphi(x) = sin(x)$

Conditions:

$$w(0,t) = 0 \qquad w(\pi,t) = 0 \qquad w_x(0,t) = e^{-t} \qquad w_x(\pi,t) = -e^{-t}$$

$\phi(m,r)$ and $\phi(m)$ they are continuous.

we take n=5 moments and is approaching $\varphi(x) = w(x,0)$ where the accuracy is

$$\int_0^{\pi} \left(p_{15}(x) - \varphi(x)\right)^2 dtdx = 0.00738463$$

In the Fig.1 the graphics of: $p_{15}(x)$ (magenta color) and $\varphi(x)$ (light blue color) are superimposed.

we take $n = 9$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^{\pi}\int_0^{\infty} \left(p_{29}(x) - w(x,t)\right)^2 dtdx = 0.0710157$$

In the Fig. 2 the graphics of: $p_{29}(x)$(magenta color) and $\varphi(x)$ (light blue color) are superimposed.

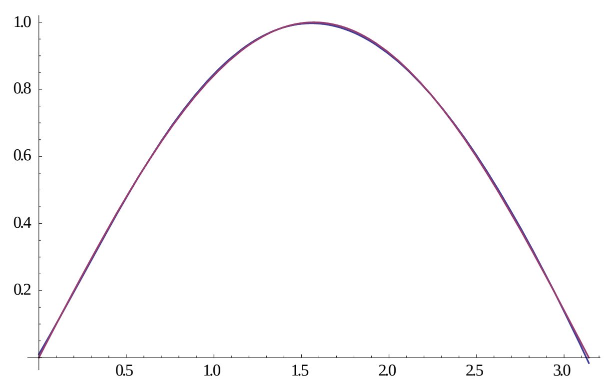

Fig. 1: $p_{15}(x)$ and$\varphi(x)$ example 1

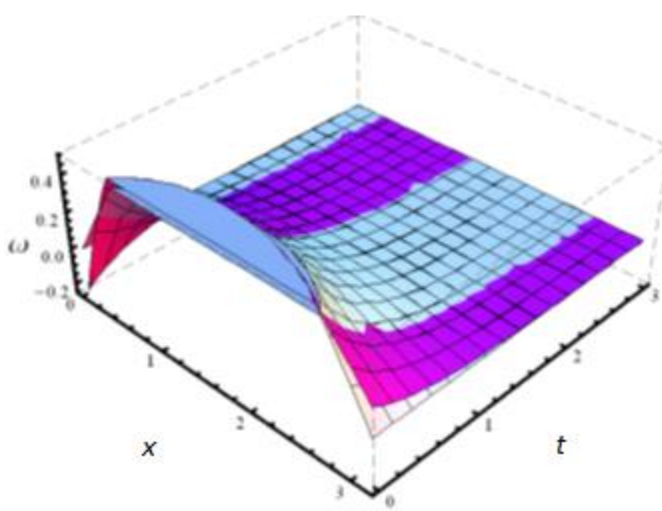

Fig. 2:$w(x,t)$ and $p_{29}(x,t)$ example 1

Example 2

We consider the equation

$$w_t - w_{xx} = \frac{1}{4} e^{-\left(\frac{x}{2}+\frac{t}{4}\right)} \left(-2 + e^{\frac{x}{2}}\right) \ \ in \quad 1 < x < 3; \qquad t > 0$$

whose solution is: $w(x,t) = e^{-\left(\frac{x}{2}+\frac{t}{4}\right)} - e^{-\frac{t}{4}}$ $\qquad \varphi(x) = e^{-\left(\frac{x}{2}\right)} - 1$

Conditions:

$$w(1,t) = e^{-\left(\frac{1}{2}+\frac{t}{4}\right)} - e^{-\frac{t}{4}} \qquad w(3,t) = e^{-\left(\frac{3}{2}+\frac{t}{4}\right)} - e^{-\frac{t}{4}} \qquad w_x(1,t) = -\frac{1}{2} e^{-\frac{t}{4}-\frac{1}{2}}$$

$$w_x(3,t) = -\frac{1}{2} e^{-\frac{t}{4}-\frac{3}{2}}$$

$\phi(m)$ we find that it is discontinuous in some integer values.

we takea base: $\psi_i(m) = m^i \qquad i = 0,1,2,3.4$

and we multiply both members of (3) by $\psi_j(m)$ and we integrate with respect to *m*.

We take n=5 moments and is approaching $\varphi(x) = w(x,0)$ where the accuracy is

$$\int_0^{\pi} \left(p_{15}(x) - \varphi(x)\right)^2 dtdx = 1.44566\text{x}10^{-6}$$

In the Fig.3 the graphics of: $p_{15}(x)$ (magenta color) and $\varphi(x)$ (light blue color) are superimposed.

$\phi(m,r)$ is continuous.We take $n = 9$ moments and is approaching $w(x,t)$where the accuracy is

$$\int_0^{\pi}\int_0^{\infty} \left(p_{29}(x) - w(x,t)\right)^2 dtdx = 0.0259608$$

In the Fig. 4 the graphics of

: $p_{29}(x)$(magenta color) and $\varphi(x)$ (light blue color) are superimposed.

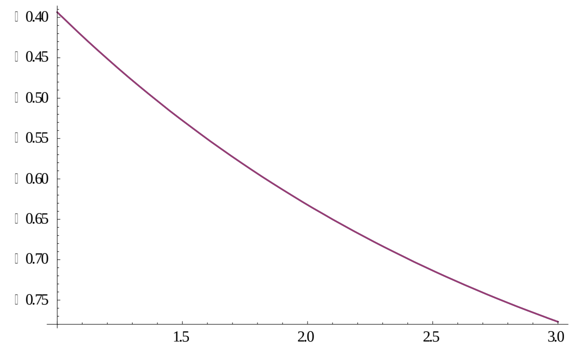

Fig. 3: $p_{15}(x)$ and$\varphi(x)$ example 2

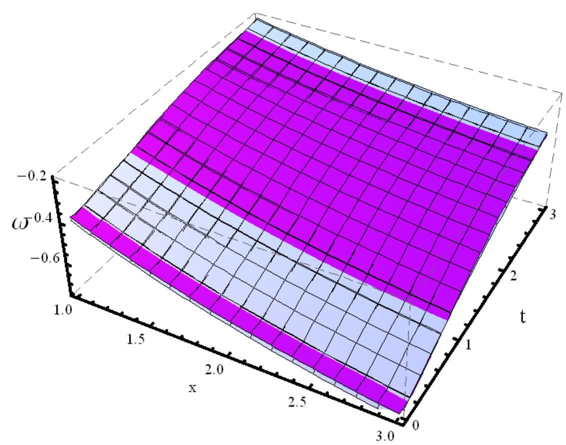

Fig. 4:$w(x,t)$ and $p_{29}(x,t)$ example 2

CONCLUSION

A parabolic equation of the form $w_t - w_{xx} = R(x,t)$ where the unknown function $w(x,t)$ is defined in $D = (a_1, b_1) \times (0, \infty)$ under Cauchy conditions with unknown initial condition $\varphi(x) = w(x,0)$ it can be solved numerically by applying inverse moment problem techniques in two steps.

1. first the partial derivatives equation is written as an integral equation

$$\int_{a_1}^{b_1} u\left(m, \left(\frac{m}{b_1}\right)^2, x, 0\right) w(x,0)dx = A\left(m, \left(\frac{m}{b_1}\right)^2\right) -$$

$$-\int_0^{\infty} u\left(m, \left(\frac{m}{b_1}\right)^2, b_1, t\right) w_x(b_1,t)dt - \int_0^{\infty} u\left(m, \left(\frac{m}{b_1}\right)^2, a_1, t\right) w_x(a_1,t)dt -$$

$$-\iint u\left(m, \left(\frac{m}{b_1}\right)^2, x, t\right) R(x,t)dA = \phi(m)$$

To solve this integral equation we give integer values to m: $m = 0,1,2, \dots n$ Then $\int_{a_1}^{b_1} \varphi(x) H_m(x) dx = \phi(m) = \mu_m$. We interpret as a generalized moments problem, $p_{1n}(x)$ is the numerical approximation with the truncated expansion method for $\varphi(x) = w(x,0)$,with

$H_m(x) = u\left(m, \left(\frac{m}{b_1}\right)^2, x, 0\right) = e^{-\frac{m}{b_1}x}$ $m = 0,1,2, \dots n$, where $n$is conveniently chosen.

2. To find an approximation of $w(x,t)$we consider:

$$\iint_D w(x,t) H_{mr}(x,t) dA = \frac{-G(m,r) + A(m,r) + \iint_D u\, R(x,t) dA}{\left(\left(\frac{m}{b_1}\right)^2 - (r+1)\right)} = \phi(m,r) = \mu_{mr}$$

where $H_{m,r}(x) = u(m,r,x,t)$. We can consider it as a two-dimensional generalized moment problem if we discretize giving m and r non-negative integer values $m = 0,1,2, \dots, n_1$; $r = 0,2, \dots, n_2$, where n_1 and n_2 are conveniently chosen. An approximation $p_{2n}(x,t)$ is found by the truncated expansion method for $w(x,t)$ where $n = n_1 \times n_2$.

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