FUZZY METRIC SPACE AND SEQUEL OF COMMON FIXED POINT THEOREM USING PROPERTY E.A.

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Abstract

In the present article, proved a common fixed point theorem in fuzzy metric space using property E.A. for a pair of weakly compatible maps.

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1. INTRODUCTION

The concept of fuzzy sets was first initiated by Zadeh [27]with a concept to delegate the vagueness in regular life laid the path to the amplification of fuzzy mathematics. Many researchers and mathematicians have developed, extended ,studied and kept up the theory of fuzzy sets and its applications, namely George and Veeramani[4,5] , Kramosil and Michalek [11], Grabiec[6],Fuller [3],Gregori and Sapena [7],Imdad, Ali and Hasan [8],Mihet [14], Sastry,Naidu and Krishn [17],Schweizer [18],Bratney and Odeh [13],Romaguera ,Sapena and Tirado [16],Shirude and Aage [21], Steimann [23], Vijayaraju and Sajath [25], Singh and Jain [22] Subrahmanyam [24], Jungck [9], Amari and Moutawakil [1], Mujahid Abbas [2], Sedghi,et.al.[19], Khan [10], Shen,et.al.[20], Wairojjana, et.al. [26] and Manthena and Manchala [12] recently proved common fixed point theorems in fuzzy metric space using property E.A.

2. PRELIMINARIES

Definition 2.1. [18]$A\ binary\ operation\ *: [0,1] \times [0,1] \rightarrow [0,1]\ is\ a\ continuous\ t-norm\ if\ for\ all\ p,q,r,s \in [0,1], the\ following\ conditions\ are\ satisfied$:

(2.11) $p * 1 = p,$

(2.1.2) $p * q = q * p,$

(2.1.3) $p * q \leq r * s\ whenever\ p \leq r\ and\ q \leq s,$

(2.14) $p * (q * r) = (p * q) * r.$

Definition 2.2 [4] *The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X × X × (0, ∞) satisfying the following conditions:*

(2.2.1) $M(x, y, t) > 0,$

(2.2.2) $M(x, y, t) = 1$ $for\ all\ t > 0\ if\ and\ only\ if\ x = y,$

(2.2.3) $M(x, y, t) = M(y, x, t),$

(2.2.4) $M(x, z, t + s) \leq M(x, y, t) * M(y, z, s),$

(2.2.5) $M(x, y, .): (0, \infty) \to (0,1]$ $is\ a\ continuous\ function, for\ all\ x, y, z \in$ X and $t, s > 0$.

Lemma 2.1. [6] $M(x, y, .)$ $is\ non\ decreasing\ for\ all\ x, y \in$ X.

Definition 2.3. [4], [7] Let $M(X, M, *)$ $be\ a\ fuzzy\ metric\ space.$

(2.3.1) A sequence $\{x_n\}$ in X is a M-Cauchy sequence if for all $\mathcal{E} \in (0,1)$, t > 0 there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \mathcal{E}$ for all n, m $\geq n_0$,

(2.3.2) A sequence $\{x_n\}$ in X is convergent to x $\in X$ if $\lim_{n \to \infty} (x_n, x, t) = 1, t > 0,$

(2.3.3) A fuzzy metric space X is M-complete if every M-Cauchy sequence in X is convergent.

Definition 2.4. [22] Two self-mappings P and Q on a fuzzy metric space *(X, M, *) are said to be compatible if* $\lim_{n \to \infty} M(PQx_n, QPx_n, t) = 1, for\ all\ t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = x \in X$.

Definition 2.5. *[22] Two self-maps P and Q of a fuzzy metric space (X, M, *) are said to be weakly compatible if they commute at their coincidence points; i.e., Px = Qx for some $x \in X$ implies that PQx = QPx.*

Remark 2.1 Two compatible self-mappings are weakly compatible, but the converse is not true.(see example 2.16 of [22]).

Definition 2.6. *[2] A Pair of self-maps (P,Q) on a fuzzy metric space (X, M, *) satisfies the property E.A. if there exists a sequence $\{x_n\}$ in X such that* $\lim_{n\to\infty} M(Px_n, x, t) = \lim_{n\to\infty} M(Qx_n, x, t) = 1$ *for some $x \in X$ and all t > 0.*

Remark 2.2 it is noted that weak compatibility and E.A. property are independent to each other. (see [15], example 2.1 and example 2.2).

Definition 2.7. *[26] A function ∅ : [0,1] → [0,1] is called an altering distance function if it satisfies the following properties:*

(2.7.1) ∅ is strictly decreasing and continuous;

(2.7.2) ∅ (λ) = 0 if and only if λ = 1.

It is obvious that $\lim_{\lambda \to 1} \emptyset(\lambda) = \emptyset(1) = 0.$

3. MAIN RESULTS

Theorem 3.1. Let $(X, M, *)$ be a fuzzy metric space and S_2 , S_1 be weakly compatible self maps of X satisfying the following property

$$\emptyset\big(M(S_2x, S_2y, t)\big) \le \alpha_1(t)\, Min\left\{\emptyset\left\{\begin{matrix}\dfrac{M(S_1x, S_2x, t)M(S_1y, S_2y, t)}{M(S_1x, S_1y, t)}, \dfrac{M(S_1x, S_2y, t)M(S_1y, S_2x, t)}{M(S_1x, S_1y, t)}, M(S_2x, S_1x, t), \\ M(S_1y, S_2y, t), M(S_1x, S_1y, t)\end{matrix}\right\}\right\}$$

$$+ \alpha_2(t)\left(\emptyset\big(M(S_2x, S_1y, 2t)\big)\right) \quad \text{... 3.1.1}$$

Where x, y ∈ X, α_1, α_2 : (0,∞) → (0,1), t > 0 and ∅ is an altering distance function. If S_2 and S_1 satisfy the property E.A. and the range of S_1 is a closed subspace X, then S_2 and S_1 have a unique common fixed point in X.

Proof: Suppose that S_2 and S_1 satisfy the property E.A., then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} S_2x_n = \lim_{n\to\infty} S_1x_n = z \in X \quad \text{--- 3.1.2}$$

Since $S_1(x)$ is a closed subspace of X. There exists u ∈ X such that

$$z = S_1u \quad \text{--- 3.1.3}$$

For x = x_n, y = u, Equation 3.1.1 becomes,

$$\emptyset\big(M(S_2x_n, S_2u, t)\big) \le \alpha_1(t)\, Min\left\{\emptyset\left\{\begin{matrix}\dfrac{M(S_1x_n, S_2x_n, t)M(S_1u, S_2u, t)}{M(S_1x_n, S_1u, t)}, \dfrac{M(S_1x_n, S_2u, t)M(S_1u, S_2x_n, t)}{M(S_1x_n, S_1u, t)}, M(S_2u, S_1x_n, t), \\ M(S_1u, S_2u, t), M(S_1x_n, S_1u, t)\end{matrix}\right\}\right\}$$

$$+ \alpha_2(t)\left(\emptyset\big(M(S_2x_n, S_1u, 2t)\big)\right)$$

Taking Limit n→∞ with using Equation 3.1.2 and Equation 3.1.3, we get

$$\emptyset\big(M(z,S_2u,t)\big) \leq \alpha_1(t)\, Min\left\{\emptyset\left\{\frac{M(z,z,t)M(z,S_2u,t)}{M(z,z,t)},\frac{M(z,S_2u,t)M(z,z,t)}{M(z,z,t)},M(S_2u,z,t),M(z,S_2u,t),M(z,z,t)\right\}\right\} + \alpha_2(t)\,\Big(\emptyset\big(M(z,z,2t)\big)\Big)$$

$$\emptyset\big(M(z,S_2u,t)\big) \leq \alpha_1(t)\, Min\big\{\emptyset\{M(z,S_2u,t),M(z,S_2u,t),M(S_2u,z,t),M(z,S_2u,t),\emptyset(1)\}\big\} + \alpha_2(t)\,\big(\emptyset(1)\big)$$

$\Rightarrow \emptyset\big(M(z,S_2u,t)\big) = 0$ which implies $M(z,S_2u,t) = 1$ i.e. $S_2u = z$. ---3.1.4

From Equation 3.1.3 and Equation 3.1.4, we have

$$S_2u = S_1u = z. \qquad \text{--- 3.1.5}$$

Since S_2, S_1 are weakly compatible, we have

$$S_2z = S_1z \qquad \text{---3.1.6}$$

Now we shall show that z is a fixed point of S_2. Suppose let us assume that $S_2z \neq z$.

In view of Equations 3.1.5, 3.1.1, with 3.1.6 and using properties of ∅, we get

$$\emptyset\big(M(S_2z,z,t)\big) = \emptyset\big(M(S_2z,S_2u,t)\big)$$

$$\leq \alpha_1(t)\, Min\left\{\emptyset\left\{\begin{matrix}\frac{M(S_1z,S_2z,t)M(S_1u,S_2u,t)}{M(S_1z,S_1u,t)},\frac{M(S_1z,S_2u,t)M(S_1u,S_2z,t)}{M(S_1z,S_1u,t)},M(S_2z,S_1z,t),\\ M(S_1u,S_2u,t),M(S_1z,S_1u,t)\end{matrix}\right\}\right\}$$

$$+ \alpha_2(t)\,\Big(\emptyset\big(M(S_2z,S_1u,2t)\big)\Big)$$

$$= \alpha_1(t)\, Min\left\{\emptyset\left\{\begin{matrix}\frac{M(S_2z,S_2z,t)M(z,z,t)}{M(S_2z,z,t)},\frac{M(S_2z,zu,t)M(z,S_2z,t)}{M(S_2z,z,t)},M(S_2z,S_2z,t),\\ M(z,z,t),M(S_2z,z,t)\end{matrix}\right\}\right\}$$

$$+ \alpha_2(t)\,\Big(\emptyset\big(M(S_2z,z,2t)\big)\Big)$$

$$= \alpha_1(t)\, Min\left\{\emptyset\left\{\frac{\emptyset(1).\emptyset(1)}{M(S_2z,z,t)},M(S_2z,z,t),\emptyset(1),\emptyset(1),M(S_2z,z,t)\right\}\right\}$$

$+ \alpha_2(t) \left(\emptyset\left(M(S_2z, z, 2t)\right)\right)$

$= \alpha_2(t) \left(\emptyset\left(M(S_2z, z, 2t)\right)\right) < \emptyset\left(M(S_2z, z, 2t)\right) < \emptyset\left(M(S_2z, z, t)\right)$, t > 0

which is a contradiction. Therefore, $S_2z = z$. Thus,

$S_2z = z = S_1z$ i.e. z is a common fixed point of S_2 and S_1 ---3.1.7

For Uniqueness, let ω∈ X be another common fixed point of S_2 and S_1 such that

$S_2\omega = S_1\omega = \omega \; and \; \omega \neq z$ ---3.1.8

Then by Equations 3.17, 3.1.8, with Equation 3.1.1 and properties of ∅, we have

$$\emptyset\left(M(z, \omega, t)\right) = \emptyset\left(M(S_2z, S_2\omega, t)\right)$$

$$\leq \alpha_1(t)\, Min \left\{ \emptyset \left\{ \begin{matrix} \frac{M(S_1z, S_2z, t)M(S_1\omega, S_2\omega, t)}{M(S_1z, S_1\omega, t)}, \frac{M(S_1z, S_2\omega, t)M(S_1\omega, S_2z, t)}{M(S_1z, S_1\omega, t)}, M(S_2z, S_1z, t), \\ M(S_1\omega, S_2\omega, t), M(S_1z, S_1\omega, t) \end{matrix} \right\} \right\}$$

$$+ \alpha_2(t) \left(\emptyset\left(M(S_2z, S_1\omega, 2t)\right)\right)$$

$$= \alpha_1(t)\, Min \left\{ \emptyset \left\{ \begin{matrix} \frac{M(z, z, t)M(\omega, \omega, t)}{M(z, \omega, t)}, \frac{M(z, \omega, t)M(\omega, z, t)}{M(z, \omega, t)}, M(z, z, t), \\ M(\omega, \omega, t), M(z, \omega, t) \end{matrix} \right\} \right\}$$

$$+ \alpha_2(t) \left(\emptyset\left(M(z, \omega, 2t)\right)\right)$$

$= \alpha_2(t) \left(\emptyset\left(M(z, \omega, 2t)\right)\right) < \emptyset\left(M(z, \omega, 2t)\right) < \emptyset\left(M(z, \omega, t)\right)$, t >0.

which is a contradiction and thus z is the unique common fixed point of S_2 and S_1.

REFERENCES

[1] Aamri, M., El Moutawakil, D. *Some new* common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002), 181-188.

[2] Abbas, M., Altun, I. and Gopal, D. *Common fixed point theorems for non compatible mappings in fuzzy metric spaces, Bulletin of Mathematical analysis and Applications 1 (2009), no. 2, 47-56.*

[3] Fuller, R.*neural fuzzy system*., Abo Akademi University, Abo, ESF Series A:443(1995).

[4] George, A., Veeramani, P. *On some results in fuzzy metric spaces,* Fuzzy sets Syst. 64 (1994), 395-399.

[5] George, A., Veeramani, P. *On some results of analysis for fuzzy metric spaces,* Fuzzy sets Syst. 90 (1997), 365-368.

[6] Grabiec, M. *Fixed points in fuzzy metric spaces,* Fuzzy sets and Systems 27(1998), no. 3, 385-389.

[7] Gregori, V., Sapena, A. *On fixed point theorems in fuzzy metric spaces,* Fuzzy Sets and Systems 125(2002), 245-252.

[8] Imdad, M., Ali J. and Hasan, M. *Common fixed point theorems in fuzzy metric spaces employing common property (E.A.),* Mathematical and Computer Modelling 55(2012), 770-778.

[9] Jungck, G. *Commuting mappings and fixed points, Amer. Math. Monthly 83(1976), 261-263.*

[10] Khan, MS., Swaleh, M. and Sessa, S. *Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30(1984), 1-9.*

[11] Kramosil, I., Michalek, J. *Fuzzy metric and statistical metric spaces,* Kybernetika 11(1975), 336-344.

[12] Manthena, P. and Manchala, R., Common fixed point theorems in fuzzy metric spaces using property E.A.,NTMSCT, Vol.,6,No. 3 (2018) 174-180.

[13] Mc Bratney, A., Odeh, IOA. *Application of fuzzy sets in soil science: fuzzy logic, fuzzy measurements and fuzzy decisions, Geoderma 77(1997), 85-113.*

[14] Mihet, D. *Fixed point theorems in fuzzy metric spaces using property E.A.,* Nonlinear Analysis 73(2010), no. 1, 2184-2188.

[15] Pathak, H. K., Lopez, R. R. and Verma, R. K. *A common fixed point theorem using Implicit Relation and Property(E.A.) in Metric Spaces, Filomat 21(2007), no. 2, 211-234.*

[16] Romaguera, S., Sapena, A., Tirado, P. *The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words, Topol. Appl. 154(2007), 2196-2203.*

[17] Sastry, K. P. R., Naidu, G. A. and Marthanda Krishna, K. *Common fixed point theorems for four self maps on a fuzzy metric space satisfying common E.A. Property,* Advances in Applied Science Research 6(2015), no. 10, 35-39.

[18] Schweizer, B., Sklar, A. *Statistical metric spaces*, Pacific J. Math. 10(1960), no. 1, 313-334.

[19] Sedghi, S., Shobe, N. and Aliouche, A. *A common fixed point theorem for weakly compatible mappings in fuzzy metric spaces,* General Mathematics 18(2010), no. 3, 3-12.

[20] Shen, Y., Qiu, D. and Chen, W. *Fixed point theorems in fuzzy metric spaces.* Applied Mathematics Letters 25(2012), no. 2, 138-141.

[21] Shirude, M.T., Aage, C.T. *Some Fixed Point Theorems using Property E.A. in Fuzzy Metric Spaces,* IJESC (2016), no 11, 3411-3414.

[22] Singh, B., Jain, S. *Semicompatibility and fixed point theorems in fuzzy metric space using implicit relation,* International Journal of Mathematics and Mathematical Sciences 2005(2005), 2617-2629.

[23] Steimann, F. *On the use and usefulness of fuzzy sets in medical AI,* Artificial Intelligence in Medicine 21(2001), 131-137.

[24] Subrahmanyam, P. V. *A common fixed point theorem in fuzzy metric spaces,* Inform. Sci 83(1995), no. 2, 109-112.

[25] Vijayaraju, P., Sajath, Z.M.I. *Some common fixed point theorems in fuzzy metric spaces,* International Journal of Mathematical Analysis 3(2009), no. 15, 701-710.

[26] Wairojjana, N., Dosenovi c, T., Raki c, D., Gopal, D. and Kumam, P. *An altering distance function in fuzzy metric fixed point theorems,* Fixed Point Theory and Applications(2015), 2015:69.

[27] Zadeh, L. A. *Fuzzy sets,* Inf. Control 8(1965), 338-353.