GROWTH OF SOLUTIONS FOR A DELAYED KIRCHHOFF-TYPE VISCOELASTIC EQUATION

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Abstract

In this paper, we deal with a delayed Kirchhoff-type viscoelastic equation. Under suitable conditions, we establish the growth of solution.

Keywords: *Distributed delay, growth, Kirchhoff-type, viscoelastic equation.*

1. INTRODUCTION

In this paper, we consider the following delayed Kirchhoff-type viscoelastic equation

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t \varpi(t-q)\Delta u(q)dq \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)| u_t(x,t-q)dq \\ = b|u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in [0,\infty), \\ u_t(x,-t) = f_0(x,t), & (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $b,\ \mu_1$ are positive constants, $p > 2$, and τ_1, τ_2 are time delay with $0 \le \tau_1 < \tau_2$ and μ_2 is bounded function and ϖ is a differentiable function. $M(s)$ is a non-negative function of C^1 for $s \ge 0$, satisfies $M(s) = m_0 + \alpha s^\gamma, m_0 > 0, \alpha \ge 0$ and $\gamma \ge 0$, specially we take $M(s) = 1 + s^\gamma$ where $m_0 = 1, \alpha = 1$.

Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine (Kafini and Messaoudi (2016)). The problem (1) is a general form of a model introduced by Kirchhoff (1883). To be more precise, Kirchhoff recommended a model denoted by the equation for $f = g = 0$,

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{2}$$

for $0 < x < L,\ t \geq 0$, where $u(x,t)$ is the lateral displacement, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length, ρ_0 is the initial axial tension, δ is the resistance modulus, and f and g are the external forces. Furthermore, (2) is called a degenerate equation when $\rho_0 = 0$ and nondegenerate one when $\rho_0 > 0$.

In 1986, Datko et al. indicated that delay is a source of instability. Nicaise and Pignotti (2006) considered the following wave equation with a linear damping and delay term

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0. \tag{3}$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. In the absence of delay, Zuazua (1990) looked into exponentially stability for the equation (3).

Wu and Tsai [24] (2006), considered the following Kirchhoff-type equation

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + |u_t|^{r-2}u_t = |u|^{p-2}u, \tag{4}$$

with the positive upper bounded initial energy and they obtained the blow-up of solutions for the equation (4). Ye (2013), considered the global existence results by constructing a stable set in $H_0^1(\Omega)$ and showed the decay by using a lemma of Komornik for the nonlinear Kirchhoff-type equation (4) with dissipative term.

When $M(s) = 1$, the equation (1) becomes the following form

$$u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, u_t(x, t-\rho)d\rho$$

$$= b|u|^{p-2}u. \tag{5}$$

In [3], Choucha (2020) et al. obtained the blow-up of solutions under appropriate conditions of the equation (5). In [4], Choucha (2020) et al. showed the exponential growth of solution for the equation (5).

In [23], Yüksekkaya and Pişkin (2021) proved the nonexistence of global solutions of the equation (1). Our goal in this paper is to get the growth of solutions for the Kirchhoff-type viscoelastic equation (1) with distributed delay. In recent years, some other authors investigate hyperbolic type equations with delay term (see Antontsev et al. (2021), Choucha et al. (2021), Doudi and Boulaaras (2020), Pişkin and Yüksekkaya (2020), Pişkin and Yüksekkaya (2021a), (2021b), (2021c), (2021d), (2021e), (2021f), (2021g), Yüksekkaya et al. (2021a), (2021b), Yüksekkaya and Pişkin (2021c)).

The paper is organized as follows: In section 2, we give some materials that will be used later. In section 3, we state and prove our main result.

2. PRELIMINARIES

In this part, we denote some materials for the proof of our result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Now, we give some assumptions used in this paper:

(A1) $\varpi \in (R_+, R_+)$ is decreasing function, such that

$$\varpi(t) \geq 0, 1 - \int_0^\infty \varpi(q)\,dq = l > 0. \tag{6}$$

(A2) There exists a constant $\xi > 0$, such that

$$\varpi'(t) \leq -\xi\varpi(t), t \geq 0. \tag{7}$$

(A3) $\mu_2\colon [\tau_1, \tau_2] \to R$ is bounded function, so that

$$\left(\frac{2\delta-1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_2(q)|\,dq \leq \mu_1,\ \ \delta > \frac{1}{2}. \tag{8}$$

Let $B_p > 0$ be the constant satisfying (Adams (2003), Pişkin (2017))

$$\|\upsilon\|_p \leq B_p\|\nabla\upsilon\|_p,\ \ for\ \upsilon \in H_0^1(\Omega). \tag{9}$$

It holds

$$\int_0^t \varpi(t-q)\big(\nabla u(q), \nabla u_t(t)\big)\,dq = -\frac{1}{2}\varpi(t)\|\nabla u(t)\|^2 + \frac{1}{2}(\varpi' o \nabla u)(t)$$
$$-\frac{1}{2}\frac{d}{dt}\left[(\varpi o \nabla u)(t) - \left(\int_0^t g(q)\,dq\right)\|\nabla u(t)\|^2\right], \tag{10}$$

where

$$(\varpi o \nabla u)(t) = \int_\Omega \int_0^t \varpi(t-q)|\nabla u(t) - \nabla u(q)|^2\,dq. \tag{11}$$

Firstly, similar to Nicaise and Pignotti (2008), we introduce the new variable

$$y(x, \rho, q, t) = u_t(x, t - q\rho),$$

then, we get

$$\begin{cases} qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0 \\ y(x,0,q,t) = u_t(x,t). \end{cases} \tag{12}$$

Hence, the problem (1) is equivalent to:

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t \varpi\,(t-q)\Delta u(q)dq \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)||y(x,1,q,t)|dq \\ = b|u|^{p-2}u, \qquad x \in \Omega,\ t > 0, \\ qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0, \end{cases} \tag{13}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \qquad x \in \partial\Omega, \\ y(x,\rho,q,0) = f_0(x,q\rho), \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), \end{cases} \tag{14}$$

where

$$y(x,\rho,q,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

We give the local existence without proof similar to (Georgiev and Todorova (1994), Pişkin (2015), Wu and Tsai [25] (2006)).

Theorem 1 Suppose that (6), (7) and (8) hold. Let

$$\begin{cases} 2 \leq p, \qquad if\ n = 1,2, \\ 2 < p < \frac{2(n-1)}{n-2},\ \ if\ n \geq 3. \end{cases} \tag{15}$$

Then, for any initial data

$$(u_0,u_1,f_0) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega) \times L^2\left(\Omega \times (0,1) \times (\tau_1,\tau_2)\right),$$

with compact support, then the problem (13)-(14) has a unique solution

$$u \in C\left([0,T]; \left(H_0^1(\Omega) \cap H^2(\Omega)\right)\right) \times H_0^1(\Omega) \times L^2\left(\Omega \times (0,1) \times (\tau_1,\tau_2)\right),$$

for some $T > 0$.

Now, we give the global existence result without proof similar to (Wu and Tsai [25] (2006)).

Theorem 2 Assume that (6), (7), (8) and (15) hold. If $u_0 \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)$, $u_1 \in H_0^1(\Omega)$ and

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{16}$$

where C_* is the Poincare's constant. Then, the local solution $u(t,x)$ is global in time.

Lemma 3 Suppose that (6), (7), (8) and (15) hold. Let $u(t)$ be a solution of (13), then $E(t)$ is nonincreasing, such that

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\left(1-\int_0^t \varpi(q)\,dq\right)\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}$$

$$+\frac{1}{2}(\varpi o\nabla u)(t)+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx-\frac{b}{p}\|u\|_p^p, \tag{17}$$

satisfies

$$E'(t)\le -c_1\left(\mu_1\|u_t\|^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx\right). \tag{18}$$

Proof. Multiplying the first equation of (13) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|^2+\frac{1}{2}\left(1-\int_0^t \varpi(q)\,dq\right)\|\nabla u\|^2\right.$$

$$\left.+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2}(\varpi o\nabla u)(t)-\frac{b}{p}\|u\|_p^p\right\}$$

$$=-\mu_1\|u_t\|^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(q)||y(x,1,q,t)|dqdx$$

$$+\frac{1}{2}(\varpi' o\nabla u)(t)-\frac{1}{2}\varpi(t)\|\nabla u\|^2, \tag{19}$$

and, we obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx$$

$$=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2|\mu_2(q)|\,yy_\rho dq\,d\rho dx$$

$$=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,0,q,t)|dqdx$$

$$-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx$$

$$=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\|u_t\|^2$$

$$-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx. \tag{20}$$

Thus, we have

$$\frac{d}{dt}E(t) = -\mu_1\|u_t\|^2 - \int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(q)||u_t y(x,1,q,t)|dqdx + \frac{1}{2}(\varpi' o\nabla u)(t)$$

$$-\frac{1}{2}\varpi(t)\|\nabla u\|^2 + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\|u_t\|^2$$

$$-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx. \tag{21}$$

From (19) and (20), we obtain (17). Also, utilizing Young's inequality, (6), (7) and (8) in (21), we get (18).

Now, to prove our main result, we define

$$H(t) = -E(t)$$

$$= \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t \varpi(q)dq\right)\|\nabla u\|^2 - \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}$$

$$-\frac{1}{2}(\varpi o\nabla u)(t) - \frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx. \tag{22}$$

3. GROWTH OF SOLUTIONS

In this part, we establish the growth of solutions for the problem (13)-(14).

Theorem 4 Assume that (6)-(8), (15) and (16) hold. Suppose further that $E(0) < 0$ holds. Then, the unique local solution of the problem (13) grows exponentially.

Proof. By (17), we get

$$E(t) \leq E(0) \leq 0. \tag{23}$$

Thus,

$$H'(t) = -E'(t) \geq c_1\left(\|u_t\|^2 + \int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx\right)$$

$$\geq c_1\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx \geq 0, \tag{24}$$

and

$$0 \le H(0) \le H(t) \le \frac{b}{p}\|u\|_p^p. \tag{25}$$

Set

$$K(t) = H(t) + \varepsilon \int_\Omega uu_t\, dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2\, dx, \tag{26}$$

where $\varepsilon > 0$ to be specified later.

By multiplying the first equation of (13) by u and with a derivative of (26), we get

$$K'(t) = H'(t) + \varepsilon\|u_t\|^2 + \varepsilon\int_\Omega \nabla u \int_0^t \varpi(t-q)\nabla u(q)dqdx - \|\nabla u\|^2$$

$$-\|\nabla u\|^{2(\gamma+1)} + \varepsilon b\int_\Omega |u|^p\, dx - \varepsilon\int_\Omega \int_{\tau_1}^{\tau_2}|\mu_2(q)||uy(x,1,q,t)|dqdx. \tag{27}$$

By using

$$\varepsilon\int_\Omega \int_{\tau_1}^{\tau_2}|\mu_2(q)||uy(x,1,q,t)|dqdx$$

$$\le \varepsilon\left\{\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\|u\|^2 + \frac{1}{4\delta_1}\int_\Omega \int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx\right\}, \tag{28}$$

and

$$\varepsilon\int_0^t \varpi(t-q)\, dq\int_\Omega \nabla u\nabla u(q)\, dxdq$$

$$= \varepsilon\int_0^t \varpi(t-q)\, dq\int_\Omega \nabla u\big(\nabla u(q) - \nabla u(t)\big)\, dxdq + \varepsilon\int_0^t \varpi(q)\, dq\|\nabla u\|^2$$

$$\ge \frac{\varepsilon}{2}\int_0^t \varpi(q)dq\, \|\nabla u\|^2 - \frac{\varepsilon}{2}(\varpi o\nabla u)(t). \tag{29}$$

and, from (27), we get

$$K'(t) \ge H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\left(1 - \frac{1}{2}\int_0^t \varpi(q)dq\right)\|\nabla u\|^2$$

$$-\varepsilon\|\nabla u\|^{2(\gamma+1)} + \varepsilon b\|u\|_p^p - \varepsilon\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\|u\|^2$$

$$-\frac{\varepsilon}{4\delta_1}\int_\Omega \int_{\tau_1}^{\tau_2}|\mu_2(q)||y^2(x,1,q,t)|dqdx + \frac{\varepsilon}{2}(\varpi o\nabla u)(t). \tag{30}$$

Hence, by using (24) and by setting δ_1 such that, $\frac{1}{4\delta_1 c_1} = \kappa$, substituting in (30), we obtain

$$K'(t) \geq [1 - \varepsilon\kappa]H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\left[\left(1 - \frac{1}{2}\int_0^t \varpi(q)dq\right)\right]\|\nabla u\|^2$$

$$-\varepsilon\|\nabla u\|^{2(\gamma+1)} + \varepsilon b\|u\|_p^p - \frac{\varepsilon}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\|u\|^2 + \frac{\varepsilon}{2}(\varpi o \nabla u)(t). \quad (31)$$

By (22), for $0 < a < 1$,

$$\varepsilon b\|u\|_p^p = \varepsilon p(1-a)H(t) + \frac{\varepsilon p(1-a)}{2}\|u_t\|^2 + \varepsilon ba\|u\|_p^p$$

$$+\frac{\varepsilon p(1-a)}{2}\left(1 - \int_0^t \varpi(q)dq\right)\|\nabla u\|^2$$

$$+\frac{\varepsilon p(1-a)}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{\varepsilon}{2}p(1-a)(\varpi o \nabla u)(t)$$

$$+\frac{\varepsilon p(1-a)}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\, d\rho dx. \quad (32)$$

Substituting in (31), we obtain

$$K'(t) \geq [1 - \varepsilon\kappa]H'(t) + \varepsilon\left[\frac{p(1-a)}{2} + 1\right]\|u_t\|^2$$

$$+\varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1 - \int_0^t \varpi(q)dq\right) - \left(1 - \frac{1}{2}\int_0^t \varpi(q)dq\right)\right]\|\nabla u\|^2$$

$$+\varepsilon\left(\frac{p(1-a)}{2(\gamma+1)} - 1\right)\|\nabla u\|^{2(\gamma+1)} + \varepsilon p(1-a)H(t)$$

$$-\frac{\varepsilon}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\right)\|u\|^2 + \varepsilon ba\|u\|_p^p$$

$$+\frac{\varepsilon p(1-a)}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\, d\rho dx$$

$$+\frac{\varepsilon}{2}(p(1-a)+1)(\varpi o \nabla u)(t). \quad (33)$$

By utilizing Poincare's inequality, we have

$$K'(t) \geq [1 - \varepsilon\kappa]H'(t) + \varepsilon\left[\frac{p(1-a)}{2} + 1\right]\|u_t\|^2 + \frac{\varepsilon}{2}(p(1-a)+1)(\varpi o \nabla u)(t)$$

$$+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\int_0^t \varpi(q)dq\left(\frac{p(1-a)-1}{2}\right)\right.$$

$$\left.-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)\right\}\|\nabla u\|^2+\varepsilon ba\|u\|_p^p$$

$$+\varepsilon\left(\frac{p(1-a)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)}+\varepsilon p(1-a)H(t)$$

$$+\frac{\varepsilon p(1-a)}{2}\int_\Omega \int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx. \quad (34)$$

Choosing $a > 0$ small enough, such that

$$\alpha_1=\frac{p(1-a)}{2}-1>0,$$

$$\frac{p(1-a)}{2(\gamma+1)}-1>0,$$

and

$$\int_0^\infty \varpi(q)dq<\frac{\frac{p(1-a)}{2}-1}{\frac{p(1-a)}{2}-\frac{1}{2}}=\frac{2\alpha_1}{2\alpha_1+1}, \quad (35)$$

then, choosing κ large enough, such that

$$\alpha_2=\left(\frac{p(1-a)}{2}-1\right)-\int_0^t \varpi(q)dq\left(\frac{p(1-a)-1}{2}\right)-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(q)|dq\right)>0.$$

Once κ and a are fixed, picking ε small enough, such that

$$\alpha_3=1-\varepsilon\kappa>0,$$

and

$$K(t)\le\frac{b}{p}\|u\|_p^p. \quad (36)$$

Hence, for some $\beta > 0$, the estimate (34) takes the form

$$K'(t)\ge\beta\{H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(\varpi o\nabla u)(t)$$

$$+\|u\|_p^p+\int_\Omega \int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx\Big\}, \quad (37)$$

and

$$K(t) \geq K(0) > 0, t > 0. \tag{38}$$

Now, utilizing Young's and Poincare's inequalities, by (26), we obtain

$$K(t) = \left(H(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 \, dx\right)$$

$$\leq c\left[H(t) + \left|\int_\Omega u u_t dx\right| + \|u\|^2 + \|\nabla u\|^2\right]$$

$$\leq c[H(t) + \|\nabla u\|^2 + \|u_t\|^2]. \tag{39}$$

For $c > 0$, and since $H(t) > 0$, by (13) we get

$$-\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t \varpi(q)dq\right)\|\nabla u\|^2$$

$$-\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \frac{1}{2}(\varpi o \nabla u)(t)$$

$$-\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx + \frac{b}{p}\|u\|_p^p$$

$$> 0, \tag{40}$$

then

$$\frac{1}{2}\left(1 - \int_0^t \varpi(q)dq\right)\|\nabla u\|^2$$

$$< \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + \|\nabla u\|^{2(\gamma+1)}$$

$$+(\varpi o \nabla u)(t) + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx. \tag{41}$$

Therefore,

$$\|\nabla u\|^2 < \frac{2b}{p}\|u\|_p^p + 2(\varpi o \nabla u)(t) + \left(\int_0^t \varpi(q)dq\right)\|\nabla u\|^2$$

$$+2\|\nabla u\|^{2(\gamma+1)} + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx. \tag{42}$$

Also, by using (6), to obtain

$$\|\nabla u\|^2 < \frac{2b}{p}\|u\|_p^p + 2(\varpi o \nabla u)(t) + (1-l)\|\nabla u\|^2$$

$$+2\|\nabla u\|^{2(\gamma+1)} + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx. \quad (43)$$

As a result, inserting (43) into (39), to see that there exists $k_1 > 0$, such that, for $\forall t > 0$

$$K(t) \le k_1[H(t) + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \|u_t\|^2 + \frac{b}{p}\|u\|_p^p + (\varpi o \nabla u)(t)$$

$$+\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|\,|y^2(x,\rho,q,t)|dq\,d\rho dx\Big]. \quad (44)$$

By the inequalities (37) and (44), we get

$$K'(t) \ge \lambda K(t), \quad (45)$$

where $\lambda > 0$, depending only on β and k_1.

An integration of (45), we have

$$K(t) \ge K(0)e^{\lambda t}, \ \forall t > 0. \quad (46)$$

By (25) and (36), we get

$$K(t) \le H(t) \le \frac{b}{p}\|u\|_p^p. \quad (47)$$

From (46) and (47), we obtain

$$\|u\|_p^p \ge Ce^{\lambda t}, \forall t > 0.$$

Thus, the proof is completed.

CONCLUSIONS

Recently, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no growth of solutions for the delayed Kirchhoff-type viscoelastic equation with delay term. Under suitable conditions, we have been proved the growth of solutions.

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