SOME CURVATURE PROPERTİES OF NORMAL PARACONTACT METRIC MANIFOLDS

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Abstract

The object of the present paper is to study curvature properties of a normal paracontact metric manifold with constant sectional curvature satisfying the conditions $\tilde{Z}(\xi,X)R=0$, $\tilde{Z}(\xi,X)P=0$, $\tilde{Z}(\xi,X)\tilde{Z}=0$, $\tilde{Z}(\xi,X)S=0$ and $\tilde{Z}(\xi,X)\tilde{C}=0$. According to these cases, we classified normal paracontact metric manifolds, where R is the Riemanniann curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor, P is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor and S is the Ricci tensor.

Keywords: *Normal paracontact metric manifold, Einstein manifold, concircular curvature tensor.*

1. INTRODUCTION

Recently paracontact geometry has become a popular field of study of differential geometry. The study on paracontact geometry was started by Kenayuki and Williams [11]. Zamkovoy studied canonical connections on paracontact manifolds and found interesting results about their properties [22]. In 2009 and 2014, Welyczko studied Legendre curves and Slant curves in 3-dimensional normal almost paracontact metric manifolds [16, 18]. In 2015, Erken studied 3-dimensional normal almost paracontact metric manifold [10]. Atçeken et. al studied semiparallel submanifolds of a normal paracontact metric manifold in [3]. Since then, (para)contact geometry has been studied extensively by many geometers [8,9,12,17].

In this paper, we have investigated the derivative effects of the cocircular curvature tensor of a normal paracontact metric manifold on some other curvature tensors. In the second section, some basic definitions and formulas such as Ricci tensor, Ricci operator, scalar curvature function of a normal paracontact metric manifold with constant sectional curvature have been introduced. In the third section, we have studied the curvature tensors of a normal paracontact metric manifold with constant sectional curvature satisfying the conditions $\tilde{Z}(\xi,X)R=0$, $\tilde{Z}(\xi,X)P=0$, $\tilde{Z}(\xi,X)\tilde{Z}=0$, $\tilde{Z}(\xi,X)S=0$ and $\tilde{Z}(\xi,X)\tilde{C}=0$. According these cases, we classified normal paracontact metric manifolds, where R is the Riemannian curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor, P is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor and S is the Ricci tensor.

2. PRELIMINARIES

A n −dimensional differentiable manifold (M,g) is said to be an almost paracontact metric manifold if there exist on M a $(1,1)$ tensor field ϕ, a contravariant vector ξ and a 1-form η −such that

$$\phi^2X=X-\eta(X)\xi, \quad \phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1 \tag{1}$$

and

$$g(\phi X,\phi Y)=g(X,Y)\xi-\eta(X)\eta(Y), \quad \eta(X)=g(X,\xi), \tag{2}$$

for any $X,Y\in\chi(M)$. If the covariant derivative of ϕ satisfies

$$(\nabla_X\phi)Y=-g(X,Y)\xi-\eta(Y)X+2\eta(X)\eta(Y)\xi \tag{3}$$

then, M is called normal paracontact metric manifold, where ∇ is Levi-Civita connection. From (3), we can easily to see that

$$\phi X=\nabla_X\xi \tag{4}$$

for any $X\in\chi(M)$ [11].

On the other hand if such a manifold has constant sectional curvature equal to c, then it's the Riemannian curvature tensor is R given by

$$\begin{aligned} R(X,Y)Z=&\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\} \\ &+\frac{c-1}{4}\{\ \eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi \\ &-g(\phi X,Z)\phi Y-2g(\phi X,Y)\phi Z\}-g(Y,Z)\eta(X)\xi+g(\phi Y,Z)\phi X \end{aligned} \tag{5}$$

for any vector fields $X,Y,Z\in\chi(M)$.

The concircular curvature tensor, projective curvature tensor and quasi-conformal curvature tensor of a normal paracontact metric manifold M^{2n+1} are, respectively, defined by

$$\tilde{Z}(X,Y)Z=R(X,Y)Z-\frac{\tau}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{6}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{7}$$

$$\tilde{C}(X,Y)Z = a.R(X,Y)Z + b.[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y] \tag{8}$$

where a and b are two scalars, R is the Riemannian curvature tensor, τ is the scalar curvature function of M and Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$ [19,20].

Now we get some equations that we will use later.

Let M be $n-$dimensional a normal paracontact metric manifold. In (5) choosing $X = \xi$, we get

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y, \tag{9}$$

in the same way in (5) putting $Z = \xi$, we obtain

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{10}$$

Also from (10), we have

$$R(X,\xi)\xi = X - \eta(X)\xi. \tag{11}$$

Taking the inner product both of the sides (5) with $\xi \in \chi(M)$, we obtain

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y). \tag{12}$$

In the same way we obtain from (6), (7) and (8)

$$\tilde{Z}(\xi,Y)Z = \left[1 - \frac{\tau}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y], \tag{13}$$

$$\tilde{Z}(\xi,Y)\xi = \left[1 - \frac{\tau}{n(n-1)}\right][\eta(Y)\xi - Y], \tag{14}$$

$$P(\xi,Y)Z = g(Y,Z)\xi - \frac{1}{n-1}[S(Y,Z)\xi + \eta(Z)Y], \tag{15}$$

$$P(\xi,Y)\xi = \frac{1}{n-1}[\eta(Y)\xi - Y], \tag{16}$$

$$\tilde{C}(\xi,Y)Z = \left[\frac{4a+b[c(n-6)+7n-6]}{4} - \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right]\right][g(Y,Z)\xi - \eta(Z)Y], \tag{17}$$

$$\tilde{C}(\xi,Y)\xi = \left[\frac{4a+b[c(n-6)+7n-6]}{4} - \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right]\right][\eta(Y)\xi - Y]. \tag{18}$$

Also here for the orthonormal basis $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$,

$$S(X,Y) = g(R(X,e_i)e_i,Y) + g(R(X,\xi)\xi,Y), \tag{19}$$

with the help of (19), the Ricci tensor and the Ricci operator are given by

$$S(X,Y)=\left(\frac{c(n-6)+3n+2}{4}\right)g(X,Y)+\frac{c(6-n)+n-10}{4}\eta(X)\eta(Y), \quad (20)$$

and

$$Q(X)=\left(\frac{c(n-6)+3n+2}{4}\right)X+\frac{c(6-n)+n-10}{4}\eta(X)\xi. \quad (21)$$

From here we can reach the following conclusion.

Corollary 2.1. Every normal paracontact metric manifold with constant sectional curvature is an η −Einstien manifold.

Also we choosing $Y=\xi$ in (20), we obtain

$$S(X,\xi)=(n-2)\eta(X), \quad (22)$$

and from (21), we get

$$Q\xi=(n-2)\xi. \quad (23)$$

On the other hand the scaler curvature function of $M$is given by

$$\tau=\frac{(n-1)[c(n-6)+3n+4]-4}{4}. \quad (24)$$

3. A NORMAL PARACONTACT METRIC MANIFOLD SATISFYING CERTAIN CURVATURE CONDITIONS

Theorem 3.1. Let $M(c)$ be n −dimensional a normal paracontact metric manifold. Then $\tilde{Z}(\xi,X)R=0$ if and only if M either is a real space form with constant sectional curvature $c=1$ or the scalar curvature $\tau=n(n-1)$.

Proof: Let us suppose that $\tilde{Z}(\xi,X)R=0$. Here we know that

$$(\tilde{Z}(X,Y)R)(Z,U,W)=\tilde{Z}(X,Y)R(Z,U)W-R(\tilde{Z}(X,Y)Z,U)W$$
$$-R\big(Z,\tilde{Z}(X,Y)U\big)W-R(Z,U)\tilde{Z}(X,Y)W, \quad (25)$$

for all $X,Y,Z,U,W\in\chi(M)$. In (25), choosing $X=\xi$, we obtain

$$(\tilde{Z}(\xi,Y)R)(Z,U,W)=\tilde{Z}(\xi,Y)R(Z,U)W-R(\tilde{Z}(\xi,Y)Z,U)W$$
$$-R\big(Z,\tilde{Z}(\xi,Y)U\big)W-R(Z,U)\tilde{Z}(\xi,Y)W=0. \quad (26)$$

Using (13) in (26), we obtain

$$0 = \left[1 - \frac{\tau}{n(n-1)}\right][g(Y, R(Z,U)W)\xi - \eta(R(Z,U)W)Y$$

$$-g(Y,Z)R(\xi,U)W + \eta(Z)R(Y,U)W$$

$$-g(Y,U)R(Z,\xi)W + \eta(U)R(Z,Y)W$$

$$-g(Y,W)R(Z,U)\xi + \eta(W)R(Z,U)Y]. \tag{27}$$

In (27) using (9), (10) and (12), we conclude

$$\left[1 - \frac{\tau}{n(n-1)}\right][R(Y,U)W - g(U,W)Y + g(Y,W)U] = 0. \tag{28}$$

This tell us that the scalar curvature of M is $\tau = n(n-1)$. On the other hand, from (28) we have

$$R(Y,U)W = g(U,W)Y - g(Y,W)U,$$

and this implies that M is a real space form with constant sectional curvature $c = 1$.

The proves our assertion. The converse is obvious.

Theorem 3.2. Let $M(c)$ be $n-$dimensional a normal paracontact metric manifold. Then $\tilde{Z}(\xi,Y)P = 0$ if and only if M either is an Einstein manifold or the scalar curvature $\tau = n(n-1)$.

Proof: Assume that $\tilde{Z}(\xi,Y)P = 0$. Then we have

$$(\tilde{Z}(\xi,Y)P)(Z,U,W) = \tilde{Z}(\xi,Y)P(Z,U)W - P(\tilde{Z}(\xi,Y)Z,U)W$$

$$-P\big(Z,\tilde{Z}(\xi,Y)U\big)W - P(Z,U)\tilde{Z}(\xi,Y)W = 0, \tag{29}$$

for all $Y,Z,U,W \in \chi(M)$. In (29) using (13), we obtain

$$0 = \left[1 - \frac{\tau}{n(n-1)}\right][g(Y, P(Z,U)W)\xi - \eta(P(Z,U)W)Y$$

$$-g(Y,Z)P(\xi,U)W + \eta(Z)P(Y,U)W$$

$$-g(Y,U)P(Z,\xi)W + \eta(U)P(Z,Y)W$$

$$-g(Y,W)P(Z,U)\xi + \eta(W)P(Z,U)Y]. \tag{30}$$

When using (15) and (16) in (30) and putting $Z = W = \xi$ we obtain

$$\left[1 - \frac{\tau}{n(n-1)}\right]\left[\frac{1}{n-1}S(Y,U) - g(Y,U) + \frac{1}{n-1}g(Y,U)\right] = 0. \tag{31}$$

Thus we have either from (31)

$$S(Y,U) = (n-2)g(Y,U), \tag{32}$$

or

$$\tau = n(n-1). \tag{33}$$

According to the equation (32) M is an Einstein manifold. Also from (33) the scalar curvature of M is $\tau = n(n-1)$.

This proves our assertion. The converse is obvious.

Theorem 3.2. Let $M(c)$ be n −dimensional a normal paracontact metric manifold. Then $\tilde{Z}(\xi,Y)S = 0$ if and only if M either is an Einstein manifold or the scalar curvature $\tau = n(n-1)$.

Proof: Next we assume that $\tilde{Z}(\xi,Y)S = 0$. Then we know

$$-S\big(\tilde{Z}(X,Y)Z,W\big) - S\big(Z,\tilde{Z}(X,Y)W\big) = 0, \tag{34}$$

for all $X,Y,Z,W \in \chi(M)$. In (34) putting $X = \xi$, we have

$$S\big(\tilde{Z}(\xi,Y)Z,W\big) + S\big(Z,\tilde{Z}(\xi,Y)W\big) = 0. \tag{35}$$

Using (13) in (35), we obtain

$$\left[1 - \frac{\tau}{n(n-1)}\right][S(Y,W) - (n-2)g(Y,W)] = 0. \tag{36}$$

Thus M either is an Einstein manifold or its scalar curvature $\tau = n(n-1)$.

Theorem 3.4. Let $M(c)$ be n −dimensional a normal paracontact metric manifold. Then $\tilde{Z}(\xi,Y)\tilde{Z} = 0$ if and only if the scalar curvature of M is $\tau = n(n-1)$.

Proof: Assume that $\tilde{Z}(\xi,Y)\tilde{Z} = 0$, then we have

$$(\tilde{Z}(\xi,Y)\tilde{Z})(Z,U,W) = \tilde{Z}(\xi,Y)\tilde{Z}(Z,U)W - \tilde{Z}(\tilde{Z}(\xi,Y)Z,U)W$$
$$-\tilde{Z}\big(Z,\tilde{Z}(\xi,Y)U\big)W - \tilde{Z}(Z,U)\tilde{Z}(\xi,Y)W = 0, \tag{37}$$

for all $Y,Z,U,W \in \chi(M)$.

Using (13) in (37), we obtain

$$0 = \left[1 - \frac{\tau}{n(n-1)}\right]\left[g\big(Y,\tilde{Z}(Z,U)W\big)\xi - \eta(\tilde{Z}(Z,U)W)Y\right.$$
$$-g(Y,Z)\tilde{Z}(\xi,U)W + \eta(Z)\tilde{Z}(Y,U)W$$

$$-g(Y,U)\tilde{Z}(Z,\xi)W+\eta(U)\tilde{Z}(Z,Y)W$$

$$-g(Y,W)\tilde{Z}(Z,U)\xi+\eta(W)\tilde{Z}(Z,U)Y]. \quad (38)$$

From the equations (13) and (14) and with direct calculation we obtain

$$\left[1-\frac{\tau}{n(n-1)}\right]^2=0. \quad (39)$$

This proves our assertion.

Theorem 3.5. Let $M(c)$ be $n-$dimensional a normal paracontact metric manifold. Then $\tilde{Z}(\xi,X)\tilde{C}=0$ if and only if $M\left(\frac{4a-bc(n-6)+b(n-10)}{4a}\right)$ either is a real space form or the scalar curvature $\tau=n(n-1)$.

Proof: Assume that $\tilde{Z}(\xi,Y)\tilde{C}=0$, then we have

$$(\tilde{Z}(\xi,Y)\tilde{C})(Z,U,W)=\tilde{Z}(\xi,Y)\tilde{C}(Z,U)W-\tilde{C}(\tilde{Z}(\xi,Y)Z,U)W$$

$$-\tilde{C}\big(Z,\tilde{Z}(\xi,Y)U\big)W-\tilde{C}(Z,U)\tilde{Z}(\xi,Y)W=0, \quad (40)$$

for all $Y,Z,U,W\in\chi(M)$. From (13) we obtain,

$$0=\left[1-\frac{\tau}{n(n-1)}\right]\left[g\big(Y,\tilde{C}(Z,U)W\big)\xi-\eta(\tilde{C}(Z,U)W)Y\right.$$

$$-g(Y,Z)\tilde{C}(\xi,U)W+\eta(Z)\tilde{C}(Y,U)W$$

$$-g(Y,U)\tilde{C}(Z,\xi)W+\eta(U)\tilde{C}(Z,Y)W$$

$$\left.-g(Y,W)\tilde{C}(Z,U)\xi+\eta(W)\tilde{C}(Z,U)Y\right]. \quad (41)$$

Taking $U=\xi$ in the above equation and by using (17) and (18), we have

$$0=\left[1-\frac{\tau}{n(n-1)}\right]\left[\tilde{C}(Y,W)Z\right.$$

$$\left.-\left[\frac{4a+b[c(n-6)+7n-6]}{4}-\frac{r}{n}\left[\frac{a}{n-1}+2b\right]\right][g(W,Z)Y-g(Y,Z)W]\right]. \quad (42)$$

By taking $X\longrightarrow\phi X$ and $Y\longrightarrow\phi Y$ and using (8), we get either

$$\left[1-\frac{\tau}{n(n-1)}\right]$$

or

$$R(\phi X,\phi W)Z=\left[\frac{4a-bc(n-6)+b(n-10)}{4a}\right][g(\phi W,Z)\phi Y-g(\phi Y,Z)\phi W].$$

This completes our proof. The converse is obvious.

CONCLUSION

The concircular curvature tensor is the more a general form of the Riemann curvature tensor. Using this tensor, the conditions under which a normal paracontact metric manifold is reduced to an Einstein manifold or real space forms with constant sectional curvature $c=1$.

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