A FIXED POINT THEOREM SATISFYING INTEGRAL TYPE CONTRACTION IN REVISED FUZZY METRIC SPACE

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Abstract

In this paper, we analyze the existence of fixed points for mappings defined on a complete revised fuzzy metric space satisfying a contractive condition of integral type.

Keywords: *Revised fuzzy metric space, Integral type contraction, revised fuzzy contraction, fixed point*

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1. Introduction

Zadeh designed the hypothesis of fuzzy sets in 1965 [16]. Afterthat, Bose et.al introduce the concept of Fuzzy mappings and fixed point theorems [3]. Later on Altun et.al initiate the concept integral-type contractions on partial metric spaces and proved Fixed point theorems [2]. There After many authors [2-6, 12, 14] proved the various integral type fixed point results in various metric spaces.

Alexander Sostak [1] introduced the concept of "George-Veeramani Fuzzy Metrics Revised" in 2018 based on t-conorm. Later on Olga Grigorenko [11] et.al introduced "On t-conorm based Fuzzy (Pseudo) metrics", they develop the basics of the theory of CB-fuzzy (pseudo) metrics and compare them with "classic" fuzzy (pseudo) metrics [2020]. After that Tarkan Öner and Alexander Sostak [15] initiate the concept of On Metric-Type Spaces through Extended t-Conorms.

In 2021, Muraliraj. A & Thangathamizh. R [7] introduce the fixed point theorems based on t-conorm in Revised fuzzy metric space. Later on Muraliraj. A & Thangathamizh. R [8 & 10] prove the banach and Edelstein contractions in Revised fuzzy metric space.

2. Preliminaries

Definition 2.1: [1]

"A Revised fuzzy metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a nonempty set, $\oplus$ is a continuous t-conorm and μ is a Revised fuzzy set on $X \times X \times (0, \infty) \rightarrow [0,1]$ satisfies the following conditions: $\forall x, y, z \in X \; and \; s, t > 0$

(RGV1) $\mu(x, y, t) < 1, \forall \, t > 0$

(RGV2) $\mu(x, y, t) = 0$ if and only if $x = y, t > 0$

(RGV3) $\mu(x, y, t) = \mu(y, x, t)$

(RGV4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$

(RGV5) $\mu(x, y, -): (0, \infty) \rightarrow [0, 1)$ is continuous.

Then μ is called a Revised fuzzy metric on X".

Definition 2.2: [3]

"Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space,

1. A sequence $\{x_n\}$ in X is said to be convergent towards a point $x \in X$ if

$lim_{n\to\infty}\mu(x, y, t) = 0 \; for \; all \; t > 0$.

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for all $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\,(x_n, x_m, t) < \epsilon$ for each $n, m \geq n_0$.

3. A Revised fuzzy metric space in which each Cauchy sequence is converges is said to be complete.

4. A Revised fuzzy metric space in which each sequence has a converging subsequence is called compact".

Lemma 2.3: [2]

"For each $u, v \in X, \mu(u, v, -)$ is non-increasing function, where $(X, \mu, \oplus)$ is a revised fuzzy metric space ".where $(X, \mu, \oplus)$ is a revised fuzzy metric space.

Definition 2.9.

Let $(X, \mu, \oplus)$ be a revised fuzzy metric space. A mapping $f : X \rightarrow X$ is called revised fuzzy contractive mapping if there exists $k \in (0, 1)$, for all distinct $x, y \in X$ and $t > 0$ such that

$$\mu(fx, fy, t) \leq k\, \mu(x, y, t)$$

Theorem 2.10. **(Revised Fuzzy Banach contraction theorem)**

Let $(X, \mu, \oplus)$ be a complete Revised fuzzy metric space in which revised fuzzy contractive sequences are Cauchy. Let $f : X \to X$ be a Revised fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.

Theorem 2.11. [2]

Let (X, d) be a complete metric space, $\in (0, 1)$, and let $T : X \to X$ be a mapping such that

for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \varphi(s)ds \leq k \int_0^{d(x,y)} \varphi(s)ds$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, non negative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)ds > 0$$

then T has a unique fixed point $z \in X$ such that for each $z \in X$, $\lim_{n \to \infty} T^n x = z$.

3. Main result

Theorem 3.1.

Let $(X, \mu, \oplus)$ be a complete revised fuzzy metric space, $\in (0, 1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{\mu(Tx,Ty,t)} \varphi(s)ds \leq k \int_0^{\mu(x,y,t)} \varphi(s)ds \tag{1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, non negative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)ds > 0 \tag{2}$$

then T has a unique fixed point $z \in X$ such that for each $z \in X$, $\lim_{n \to \infty} T^n x = z$.

Proof.

Let $x \in X$ be an arbitrary point define a sequence $x_n = T^{\,n}x$. For each integer $n \geq 1$, using (1)

$$\int_0^{\mu(x_n,x_{n+1},t)} \varphi(s)ds \leq k \int_0^{\mu(x_{n-1},x_n,t)} \varphi(s)ds$$

repeating this process n times we get

$$\int_0^{\mu(x_n,x_{n+1},t)} \varphi(s)ds \leq k^n \int_0^{\mu(x_0,x_1,t)} \varphi(s)ds$$

Taking limit $n \rightarrow \infty$ we obtained

$$\lim_n \int_0^{\mu(x_n,x_{n+1},t)} \varphi(s)ds = 0$$

which from (2) implies

$$\lim_n (\mu(x_n, x_{n+1}, t)) = 0 \tag{3}$$

Now we have to show that (x_n) is a Cauchy sequence. Suppose that (x_n) is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and sub-sequence (m_p) and (n_p) such that $m_p < n_p < m_{p+1}$ with

$$\mu(x_{m_p}, x_{n_p}, t) \leq \varepsilon \qquad \mu\left(x_{m_p}, x_{n_p-1}, t\right) < \varepsilon \tag{4}$$

by using (3), we get

$$\int_0^{\mu(x_{m_p},x_{m_{p-1}},t)} \varphi(s)ds = \int_0^{\mu\left(x_{n_p},x_{m_{p-1}},t\right)} \varphi(s)ds$$

from triangular inequality and (4),

$$\begin{aligned} \mu\left(x_{m_p-1}, x_{n_p-1}, t\right) &\leq \mu\left(x_{m_p-1}, x_{m_p}, t\right) \oplus \mu\left(x_{m_p}, x_{n_p-1}, t\right) \\ &< \mu\left(x_{m_p-1}, x_{m_p}, \tfrac{t}{2}\right) \oplus \mu\left(x_{m_p-1}, x_{n_p-1}, \tfrac{t}{2}\right) \end{aligned} \tag{5}$$

Hence

$$\lim_n \int_0^{\mu(x_{m_p-1},x_{n_p-1},t)} \varphi(s)ds < \int_0^{\varepsilon} \varphi(s)ds \tag{6}$$

By using (1),(3) and (6) we have

$$\int_0^{\varepsilon} \varphi(s)ds \le \int_0^{\mu(x_{m_p}, x_{n_p}, t)} \varphi(s)ds$$

$$< k \int_0^{\mu(x_{m_{p-1}}, x_{n_{p-1}}, t)} \varphi(s)ds$$

$$< \int_0^{\varepsilon} \varphi(s)ds$$

which is a contraction so (x_n) is a Cauchy sequence. Since X is complete so there exist $z \in X$ such that $x_n \to z$. Now, using (1) and taking $n \to \infty$ we get,

$$\int_0^{\mu(Tz, x_{n+1}, t)} \varphi(s)ds \le k \int_0^{\mu(z, x_n, t)} \varphi(s)ds$$

$$< k \int_0^{\mu(z, z, t)} \varphi(s)ds = 0$$

which implies that $\mu(Tz, z, t) = 0$ and hence $Tz = z$, that is, z is fixed point of T. For uniqueness let us suppose that y and z are two distinct fixed points of T then from (1)

$$\int_0^{\mu(y, z, t)} \varphi(s)ds = \int_0^{\mu(Ty, Tz, t)} \varphi(s)ds$$

$$\le k \int_0^{\mu(y, z, t)} \varphi(s)ds$$

since $k < 1$ so $\mu(y, z, t) = 0$. Which implise $\mu(y, z, t) = 0$ or $y = z$.

Remark 3.2.

The Theorem 2.10 is a special case of our main result. That is by putting $\varphi(s) = 1$ in the inequality (1) for each $t \ge 0$, we get the inequality (2.9). The example 3.7 shows the generality of our main result.

Remark 3.3.

We have used the idea of integral Revised fuzzy contraction to generalize Revised Fuzzy Banach contraction theorem, but in a similar way we can generalize other results also related to contractive conditions of some kind, such as the ones contained in [2, 3, 4, 10, 12, 13].

Remark 3.4.

Theorem 3.1 is not true if we take zero value almost everywhere near zero for the mapping φ, following example shows our claim. In a similar way, Theorem 3.1 is not true for the negative value of φ, as in Example 3.5.

Example 3.5.

Let $X = \boldsymbol{N}$ with revised fuzzy metric defined by $\mu(x,y,t) = \frac{|x-y|}{1+|x-y|}$. Let $T: X \to X$ and $\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be defined by

$$T(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases} \qquad \varphi(s) = \begin{cases} 1 - e^{\frac{1}{1-s}} & s \geq 1 \\ 0 & s \in [0,1] \end{cases}$$

Now, since for every $x, y \in X$ and $t \geq 0$, $\mu(Tx, Ty, t) \leq 1$, hence for arbitrary $k \in (0,1)$

$$\int_0^1 \mu(Tx,Ty,t)\,\varphi(s)\ ds \leq \int_0^1 \varphi(s)\ ds = 0 \leq k \int_0^{\mu(x,y,t)} \varphi(s)\ ds$$

Thus (1) satisfied for all $k \in (0,1)$, but T has no fixed point.

Example 3.6.

Let $X = \boldsymbol{R}^+$ with revised fuzzy metric defined by $\mu(x,y,t) = \frac{|x-y|}{1+|x-y|}$. Let $T: X \to X$ and $\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be defined by $T(x) = x + 1$ and $\varphi(s) = 1$. then for an arbitrary $k \in (0,1)$

$$\int_0^{\mu(Tx,Ty,t)} \varphi(s)\ ds = -\mu(Tx,Ty,t)$$

$$= -\mu(x,y,t)$$

$$\leq -k\ \mu(x,y,t)$$

$$= k \int_0^{\mu(x,y,t)} \varphi(s)\ ds$$

Thus (1) satisfied for all $k \in (0,1)$, but T has no fixed point.

Example 3.7

Let $X = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$ with revised fuzzy metric by $\mu(x,y,t) = \frac{|x-y|}{1+|x-y|}$. Define a map $T: X \to X$ by

$$T(x) = \begin{cases} \dfrac{n}{1+n} & x = \dfrac{1}{n}, n \in N \\ 0 & x = 0 \end{cases}$$

then T is a integral revised fuzzy contraction with $\varphi(s) = s^{\frac{1}{s}-2}[1 - \log s]$ and $k = \frac{1}{2}$.

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