APPLICATION OF COMMON FIXED POINT THEOREM ON REVISED FUZZY METRIC SPACE

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Abstract

In this article, we introduces the concept of the common fixed point theorem for three function in the revised fuzzy metric space for different applications in the revised fuzzy 2-metric space and revised fuzzy 3-metric with examples. Here, too, the result is generalized and improved.

Keywords: *Revised fuzzy metric space, complete, continuous, fixed point*

1. INTRODUCTION

Alexander Sostak [1] introduced the concept of "George-Veeramani Fuzzy Metrics Revised" in 2018 based on t-conorm. Later on Olga Grigorenko [7], Juan jose Minana, Alexander Sostak, Oscar Valero introduced "On t-conorm based Fuzzy (Pseudo) metrics" [2020]. In 2020, Alexander Sostak and Tarkan Öner [9] initiate the concept of On Metric-Type Spaces Based on Extended T-Conorms.

Then Muraliraj.A and Thangathamizh.R [3] first introduce the existence of fixed point sets in the fuzzy metric space, which were revised in 2021 based on tconorm. Muraliraj.A and Thangathamizh.R [4 & 6] later prove the Banach and Edelstein contractions in the revised fuzzy metric space. In 2021, Muraliraj.A and Thangathamizh.R [5] introduce the concept of Revised fuzzy modular metric.

2. PRELIMINARIES

To initiate the concept of Revised fuzzy metric space, which was introduced by Alexander Sostak [1] in 2018 is recalled here.

Definition 2.1: [7] A binary operation $\oplus: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-conorm if it satisfies the following conditions:

a) $\oplus$ is associative and commutative,

b) $\oplus$ is continuous,

c) $a \oplus 0 = a$ for all $a \in [0, 1]$,

d) $a \oplus b \leq c \oplus d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$

Examples 2.2: [7]

i.Lukasievicz t-conorm: $a \oplus b = max\{a , b\}$
ii.Product t-conorm: $a \oplus b = a + b - ab$
iii.Minimum t-conorm: $a \oplus b = min(a + b, 1)$

Definition 2.3: [1] A Revised fuzzy metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a nonempty set, $\oplus$ is a continuous t-conorm and $\mu$is a fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions:

$\forall x, y, z \in X \, and \, s, t > 0$

(RGV1) $\mu(x, y, t) < 1 , \forall t > 0$

(RGV2) $\mu(x, y, t) = 0$ if and only if $x = y, t > 0$

(RGV3) $\mu(x, y, t) = \mu(y, x, t)$

(RGV4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$

(RGV5) $\mu(x, y, -)$: $(0, \infty) \to [0, 1)$ is continuous.

Then μ is called a Revised fuzzy metric on X.

Example 2.4: [1] Let (X, d) be a metric space. Define $a \oplus b = max\{a, b\}$ for all $a, b \in [0, 1]$, and define $\mu : X \times X \times (0, \infty) \to [0, 1]$ as

$$\mu(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

$\forall x, y, z \in X \, and \, t > 0$. Then $(X, \mu, \oplus)$ is a Revised fuzzy metric space.

Definition 2.5: [1] Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space, for $t > 0$ the open ball $B(x, r, t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : \mu(x, y, t) < r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $\mathrm{B}(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the Revised fuzzy metric μ.

Definition 2.6: [3] Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space,

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$lim_{n\to\infty}\mu(x, y, t) = 0 \; for \; all \; t > 0.$$

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\,(x_n, x_m, t) < \epsilon$ for each $n, m \geq n_0$

3. A Revised fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

4. A Revised fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Lemma 2.7: [3] Let $(X, \mu, \oplus)$ be a Revised fuzzy metric space. For all $u, v \in X, \mu(u, v, -)$ is non-increasing function.

Definition 2.8: A function μ is continuous in revised fuzzy metric space iff whenever $x_n \to x$, $y_n \to y$ then

$$lim_{n\to\infty}\mu(x_n, y_n, t) = \mu(x, y, t)$$

For each $t > 0$.

Definition 2.9: Two mappings A and S on revised fuzzy metric space X are weakly commuting iff,

$$\mu(ASu, SAu, t) \leq \mu(Au, Su, t)$$

For all $u \in X$ and $t > 0$.

Definition 2.10 A binary operation $\oplus: [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-conorm if $([0,1], \oplus)$ is an abelian topological monoid with unit 0 such that $a_1 \oplus b_1 \oplus c_1 \oplus d_1 \leq a_2 \oplus b_2 * c_2 \oplus d_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in$[0, 1]$.

Definition 2.11: A Revised fuzzy 2-metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a nonempty set, $\oplus$ is a continuous t-conorm and μ is a revised fuzzy set on $X^3 \times (0, \infty) \to [0, 1]$ satisfies the following conditions: $\forall x, y, z, u \in X \; and \; t_1, t_2, t_3 > 0$

(RGV1) $\mu(x, y, z, t) < 1, \forall t > 0$

(RGV2) $\mu(x, y, z, t) = 0,\; t > 0$ and when at least two of the three points are equal

(symmetry about three variables)

(RGV3) $\mu(x, y, z, t) = \mu(x, z, y, t) = \mu(y, z, x, t)$

(RGV4) $\mu(x, y, z, t_1 + t_2 + t_3) \leq \mu(x, y, u, t_1) \oplus \mu(x, u, z, t_2) \oplus \mu(u, y, z, t_3)$

(This corresponds to tetrahedron inequality in 2-metric space).

This function t value $\mu(x, y, z, t)$ may be interpreted as the probability that area of triangle is less than t.

(RGV5) $\mu(x, y, z, -): (0, \infty) \rightarrow [0, 1)$ is right continuous.

Then μ is called a Revised fuzzy 2-metric on X.

Definition 2.12: Let $(X, \mu, \oplus)$ be a Revised fuzzy 2-metric space,

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$lim_{n\to\infty}\mu(x_n, x, a, t) = 0 \text{ for all } a \in X \text{ and } t > 0.$$

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if

$lim_{n\to\infty}\mu(x_{n+p}, x_n, a, t) = 0$ for all $a \in X$ and $t > 0$, and $p > 0$.

3. A Revised fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

4. A Revised fuzzy 2-metric space in which every sequence has a convergent subsequence is said to be compact.

Definition 2.13: A function μ is continuous in revised fuzzy 2-metric space iff whenever $x_n \rightarrow x,\ y_n \rightarrow y$ then

$$lim_{n\to\infty}\mu(x_n, y_n, a, t) = \mu(x, y, a, t)$$

For all $a \in X$ and $t > 0$.

Definition 2.14: Two mappings A and S on revised fuzzy 2-metric space X are weakly commuting iff,

$$\mu(ASu, SAu, a, t) \leq \mu(Au, Su, a, t)$$

For all $u, a \in X$ and $t > 0$.

Definition 2.15: A binary operation $\oplus: [0,1]^4 \rightarrow [0,1]$ is called a continuous t-conorm if $([0,1], \oplus)$ is an abelian topological monoid with unit 0 such that $a_1 \oplus b_1 \oplus c_1 \oplus d_1 \leq$

$a_2 \oplus b_2 * c_2 \oplus d_2$ whenever $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2, d_1 \le d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in $[0, 1]$.

Definition 2.16: A Revised fuzzy 3-metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a nonempty set, $\oplus$ is a continuous t-conorm and μ is a fuzzy set on $X^3 \times (0, \infty) \to [0, 1]$ satisfies the following conditions: $\forall x, y, z, u \in X \; and \; t_1, t_2, t_3 > 0$

(RGV1) $\mu(x, y, z, w, t) < 1, \forall t > 0$

(RGV2) $\mu(x, y, z, w, t) = 0$, $t > 0$ and when at least two of the three points are equal (symmetry about three variables)

(RGV3) $\mu(x, y, z, w, t) = \mu(x, w, z, y, t) = \mu(y, z, w, x, t) = \mu(z, w, x, y, t) \ldots$

(RGV4) $\mu(x, y, z, t_1 + t_2 + t_3 + t_4) \le \begin{Bmatrix} \mu(x, y, z, u, t_1) \oplus \mu(x, y, u, w, t_2) \\ \oplus \mu(x, u, z, w, t_3) \oplus \mu(u, y, z, w, t_4) \end{Bmatrix}$

(This corresponds to tetrahedron inequality in 2-metric space).

This function t value $\mu(x, y, z, w, t)$ may be interpreted as the probability that area of triangle is less than t.

(RGV5) $\mu(x, y, z, w -) : (0, \infty) \to [0, 1)$ is right continuous.

Then μ is called a Revised fuzzy 3-metric on X.

Definition 2.17: Let $(X, \mu, \oplus)$ be a Revised fuzzy 3-metric space,

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$lim_{n \to \infty} \mu(x_n, x, a, b, t) = 0 \text{ for all } a, b \in X \text{ and } t > 0.$$

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if

$lim_{n \to \infty} \mu(x_{n+p}, x_n, a, b, t) = 0$ for all $a, b \in X$ and $t > 0$, and $p > 0$.

3. A Revised fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

4. A Revised fuzzy 3-metric space in which every sequence has a convergent subsequence is said to be compact.

Definition 2.18: A function μ is continuous in revised fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$ then

$$lim_{n \to \infty} \mu(x_n, y_n, a, b, t) = \mu(x, y, a, b, t)$$

For all $a, b \in X$ and $t > 0$.

Definition 2.19: Two mappings A and S on revised fuzzy 3-metric space X are weakly commuting iff,

$$\mu(ASu, SAu, a, b, t) \leq \mu(Au, Su, a, b, t)$$

For all $u, a, b \in X$ and $t > 0$.

Lemma 2.20: Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X iff, there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and satisfy :

$$d\,(Ax, Ay) \leq \alpha\, d\,(Sx, Ty)$$

for all $x, y \in X$ and $0 < \alpha < 1$. Indeed S, T and A have a unique common fixed point.

3. MAIN RESULTS

Theorem 3.1: Let $(X, \mu, \oplus)$ be a complete Revised fuzzy metric space with the condition (RFM-6) and let S and T be continuous mappings of X in X. Then S and T have a common fixed point in X, if there exists continuous mapping A and B of X into $S(X) \cap T(X)$ which commute with S and T and

$$\mu(Ax, By, kt) \leq max\begin{Bmatrix}\mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(By, Ty, t), \\ \mu(Ax, Ty, t), \mu(Ax, By, t), \mu(Sx, By, t)\end{Bmatrix}$$

For all $x, y \in X, t > 0$ and $0 < q < 1$. Then A, B, S and T have a unique common fixed point in X.

Proof : Let x_0 be any arbitrary point in X. Construct a sequence $\{y_n\}$ in X such that $y_{2n-1} = x_{2n-1} = Ax_{2n-1}$ and $y_{2n} = Sx_{2n} = Bx_{2n+1}, n = 1, 2, 3\ldots$ This can be done by (i). By using contractive condition, we obtain,

$$\mu(y_{2n+1}, y_{2n+2}, kt) = \mu(Ax_{2n}, Bx_{2n+1}, kt)$$

$$\leq max\begin{Bmatrix}\mu(Sx_{2n}, Tx_{2n+1}, t), \mu(Ax_{2n}, Sx_{2n}, t), \mu(Bx_{2n+1}, Tx_{2n+1}, t), \\ \mu(Ax_{2n}, Tx_{2n+1}, t), \mu(Ax_{2n}, Bx_{2n+1}, t), \mu(Sx_{2n}, Bx_{2n+1}, t)\end{Bmatrix}$$

$$= max\begin{Bmatrix}\mu(y_{2n}, y_{2n+1}, t), \mu(y_{2n+1}, y_{2n}, t), \mu(y_{2n}, y_{2n+1}, t), \\ \mu(y_{2n+1}, y_{2n+1}, t), \mu(y_{2n+1}, y_{2n}, t), \mu(y_{2n}, y_{2n}, t)\end{Bmatrix}$$

$$= max\begin{Bmatrix}\mu(y_{2n}, y_{2n+1}, t), \mu(y_{2n+1}, y_{2n}, t), \\ \mu(y_{2n}, y_{2n+1}, t), 0, \mu(y_{2n+1}, y_{2n}, t), 0\end{Bmatrix}$$

$$= \mu(y_{2n}, y_{2n+1}, t).$$

That is, $\mu(y_{2n+1}, y_{2n+2}, t) \leq \mu(y_{2n}, y_{2n+1}, t)$

Similarly, we have $\mu(y_{2n}, y_{2n+1}, t) \leq \mu(y_{2n-1}, y_{2n}, t)$,

So, we get $\mu(y_{2n+2}, y_{2n+1}, kt) \geq \mu(y_{2n+1}, y_n, t)$ (3.1.1)

But $(X, \mu, \oplus)$ is complete.

Hence, there exists a point z in X such that $\{y_n\} \to z$.

Also, we have $\{Ax_{2n-2}\}$, $\{Tx_{2n-1}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\} \to z$.

Since, (A, S) is compatible of type (K) and one of the mappings is continuous, using Definition (2.10),

we get $Az = Sz$. (3.1.2)

Since $A\,(X) \subseteq T\,(X)$, there exists a point u in X such that $Az = Tu$.

Now, by contractive condition we get,

$$\mu(Az, Bu, kt) \leq max \begin{Bmatrix} \mu\,(Sz, Tu, t), \mu(Az, Sz, t), \mu(Bu, Tu, t), \\ \mu(Az, Tu, t), \mu(Az, Bu, t), \mu(Sz, Bu, t) \end{Bmatrix}$$

$$= max \begin{Bmatrix} \mu(Az, Az, t), \mu(Az, Az, t), \mu(Bu, Az, t), \\ \mu(Az, Az, t), \mu(Az, Au, t), \mu(Az, Bu, t) \end{Bmatrix}$$

$\mu(Az, Bu, kt)\} \leq \mu(Az, Bu, t)$. (3.1.3)

Thus, we get $Az = Sz = Bu = Tu$. (3.1.4)

To prove $Pz = z$, we have

$$\mu(Az, Bx_{2n+1}, kt) \leq max \begin{Bmatrix} \mu(Sz, Tx_{2n+1}, t), \mu(Az, Sz, t), \mu(Bx_{2n+1}, Tx_{2n+1}, t), \\ \mu(Az, Tx_{2n+1}, t), \mu(Az, Bx_{2n+1}, t), \mu(Sz, Bx_{2n+1}, t) \end{Bmatrix}$$

Taking limit as $n \to \infty$, we get

$$\mu(Az, z, kt) \leq max \begin{Bmatrix} \mu(Sz, z, t), \mu(Az, Sz, t), \mu(z, z, t), \\ \mu(Az, z, t), \mu(Az, z, t), \mu(Sz, z, t) \end{Bmatrix}$$

$$= max\{\mu(Az, z, t), 0, 0, \mu(Az, z, t), \mu(Az, z, t), \mu(Az, z, t)\}$$

$\mu(Az, z, kt) \leq \mu(Az, z, t)$ (3.1.5)

Hence, we have $Az = Sz = z$

So, z is a common fixed point of A and S.

Also, we get $Bu = Tu = z$ (3.1.6)

Since B and T are weakly compatible, we have $TBu = Btu$. So, from (6), we get $Tz = Bz$. (3.1.7)

Again, we get

$$\mu(Ax_{2n-2}, Bz, kt) \leq max \begin{Bmatrix} \mu(Sx_{2n-2}, Tz, t), \mu(Ax_{2n-2}, Sx_{2n-2}, t), \mu(Bz, Tz, t), \\ \mu(Ax_{2n-2}, Tz, t), \mu(Ax_{2n-2}, Bz, t), \mu(Sx_{2n-2}, Bz, t) \end{Bmatrix}.$$

$$\mu(z, Bz, kt) \leq \max \begin{Bmatrix} \mu(z, Tz, t), \mu(z, z, t), \mu(Bz, Tz, t), \\ \mu(z, Tz, t), \mu(z, Bz, t), \mu(z, Bz, t) \end{Bmatrix}$$

$$= max\{\mu(z, Bz, t), 0, 0, \mu(z, Bz, t), \mu(z, Bz, t), \mu(z, Bz, t)\}$$

$$\mu(z, Bz, kt) \leq \mu(z, Bz, t). \quad (3.1.8)$$

Therefore, we have $Tz = Bz = z$. (3.1.9)

Hence, we get that z is a common fixed point of B and T.

From (3.1.5), (3.1.8) and (3.1.9),

We get $Az = Sz = Bz = Tz = z$.

So z is a common fixed point of A, B, S, and T.

For uniqueness, let w be the another common fixed point then $Aw = Bw = Sw = Pw = w$

$$\mu(Az, Bw, kt) \leq max \begin{Bmatrix} \mu(Sz, Tw, t), \mu(Az, Sz, t), \mu(Bw, Tw, t), \\ \mu(Az, Tw, t), \mu(Az, Bw, t), \mu(Sz, Bw, t) \end{Bmatrix}$$

$$= max\{\mu(Az, Bw, t), 0, 0, \mu(Az, Aw, t), \mu(Az, Bw, t), \mu(Az, Bw, t)\}$$

$$\mu(Az, Bw, kt) \leq \mu(Az, Bw, t). \quad (3.1.10)$$

From (3.1.10), and Lemma (2.1.1), we get $Az = Bw$, this implies$Az = Aw$ Hence z is a unique fixed point.

COROLLARY 3.2

Let $(X, \mu, \oplus)$ be a complete Revised fuzzy 2- metric space with the condition (FM-6) and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X, if there exists continuous mapping A and B of X into $S(X) \cap T(X)$ which commute with S and T and

$$\mu(Ax, By, kt) \le max\begin{Bmatrix} \mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(By, Ty, t), \\ \mu(Ax, Ty, t), \mu(Ax, By, t), \mu(Sx, By, t) \end{Bmatrix}$$

For all $x, y \in X, t > 0$ and $0 < q < 1$.

Then A, B, S and T have a unique common fixed point in X.

COROLLARY 3.3

Let $(X, \mu, \oplus)$ be a complete Revised fuzzy 3-metric space with the condition (FM-6) and let S and T be continuous mappings of X in X. Then S and T have a common fixed point in X, if there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T, and

$$\mu(Ax, Ay, kt) \le max\begin{Bmatrix} \mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(Ay, Ty, t), \\ \mu(Ax, Ty, t), \mu(Ax, Ay, t), \mu(Sx, Ay, t) \end{Bmatrix}$$

For all $x, y \in X, t > 0$ and $0 < q < 1$, then A, S and T have a unique common fixed point in X.

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