# COMPARATIVE STUDY OF NUMERICAL APPROXIMATION SCHEMES FOR LAPLACIAN OPERATOR IN POLAR MESH SYSTEM ON 9-POINTS STENCIL INCLUDING MIXED PARTIAL DERIVATIVE BY FINITE DIFFERENCE METHOD

**[1]Rabnawaz Mallah; [1] Inayatullah Soomro**

*[1]Department of Mathematics,*
*Shah Abdul Latif University, KhairpurMirs,*
*Post code-66020, Pakistan.*
*Corresponding Author Email: rabnawaz.mallah48@gmail.com*

## Abstract:

In this resech paper we have discussed on two different types of discretization schmes for laplacian operator which carries nine points stencil including two pair of lines symmeical and asymmetrical from planned molecule.it is well organized and modicfied five point scheme which has been devloped on finite difference method.FDM is a method which is being used for decretization of of PDES as well as ODES.In this paper we will use two different types of scheme which are devloped by FDM on polar coordinate system and in last we will discuss on error analysis,staibility and graphically behaviour of both scheme.

**Keywords:** *Finite differences method*

## Introduction:

The discret operator of isotropic laplacian equation has a dynamic role in several fundamental problems for computational simmulation modeling.The accuracy of FDM scheme is associated to the increases of stencils and comprehand to aper order in intentional directon.The proposed molecule of FDM technique is explained on isotropic that shows the procedure about the discrtization of laplacian on polar coordinate system and which is being carried out on the explicit technique of FDM. Intimate the scientific outcomes,the performances of polar coordinate system is reflected to peg away in his anistropy. Partial differential equation is being discrtized on appropriate substitue of polar gird system with boundary value condition.The constancy,illustration and equilibrium consideration will be conceded out for proposed scheme.

## Importance of Laplacian operator

There are several schemes present which have been used for the derivation of differentiation and numerical results but discretization of laplacian operator keeps dynamic role. The laplacian operator is denoted by $\nabla^2 p$.where u shows the property of operator which is non similar infield space which characterizes Cartesian coordinate system and keeps so many physical properties and also used for solving PDES and ODES for laplacian equation just like diffusion model of heat,wave spreading, gradient curl, divergence [1].For solution of physical and mathematical models and task the laplacian operator is very significant among the above scale. Laplacian operator contributes in many fields just like classical, quantum wave instead of this it is also effected on computational or mathematical models [2].There are huge application of laplacian operator in engineering for examples electricity ,fluid flow dynamics and consistent heat condition and so many other [3]. The discrete laplacian operator has operated several different projects of geometric dispensation, editing of mesh, interpolation of shape and reconstruction of surface are the best examples [4]. The linear time independent (BVC) in an essential class of problems which is being constituted by Laplacian and Poisson equations. There are many classical and absolute gird systems where the laplacian operator is dissociable. For linear ordinary differential equation with boundary value condition the numerical techniques are often applied. Modern numerical and mathematical techniques are used for solution of problems of of higher order partial and ordinary differential approximation. Finite difference approximation have three significant types in engineering fields just like finite difference method, finite element method and finite volume method and the qualities of these methods are solving the PDES and ODES [5].

For the estimation of gradient function the finite difference technique is being used in several mathematical and computational fields as in the sense of brink gird in the processing of image which is in considerable in numerical derivation especially for first derivative. For different structure of flow in the field of scientific fluid dynamics the finite difference techniques is very valuable [6].FDM is broadly used for the access to perform with FT (Fourier transform) to carried out the BVP in the girds of FDM which are used for large amount in the problems especially in nonlinear equations. [7] in addition it is being used for second order partial differential equation in term of time independent.[8-12].

## Methodology.

**Scheme 1: Derivation has been taken from Tailor series.**

$$\nabla^2 P = \frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial^2 P}{\partial r^2} = \frac{1}{4(\Delta r)^2}\left[P_{(i+1,j+1)} + P_{(i-1,j+1)} + P_{(i+1,j-1)} + P_{(i-1,j-1)} + 2P_{(i+1,j)} + 2P_{(i-1,j)} - 8P_{(i,j)}\right]\ldots\ldots\ldots(2)$$

$$\frac{\partial^2 P}{\partial \theta^2} = \frac{1}{4(\Delta \theta)^2}\left[P_{(i+1,j+1)} + P_{(i-1,j+1)} + P_{(i+1,j-1)} + P_{(i-1,j-1)} + 2P_{(i,j-1)} + 2P_{(i,j+1)} - 8P_{(i,j)}\right]\ldots\ldots\ldots\ldots.(3)$$

$\alpha = \left[\frac{r^2.\Delta r^2.\Delta\theta^2}{2(\Delta r^2 + \Delta\theta^2.r^2)}\right]$ is a weighting factor.

$$\nabla^2 P_{(i,j)} = [\alpha].\begin{bmatrix} \frac{1}{4(\Delta r)^2}\left[P_{(i+1,j+1)} + P_{(i+1,j-1)} + P_{(i-1,j+1)} + P_{(i-1,j-1)} + 2P_{(i+1,j)} + 2P_{(i-1,j)}\right] + \\ \frac{1}{r}\frac{1}{8(\Delta r)}\left[P_{(i+1,j+1)} + P_{(i+1,j-1)} - P_{(i-1,j+1)} - P_{(i-1,j-1)} + 2P_{(i+1,j)} - 2P_{(i-1,j)}\right] + \\ \frac{1}{r^2}\frac{1}{4(\Delta\theta)^2}\left[P_{(i+1,j+1)} + P_{(i-1,j+1)} + P_{(i+1,j-1)} + P_{(i-1,j-1)} + 2P_{(i,j-1)} + 2P_{(i,j+1)}\right] \end{bmatrix} - P_{(i,j)}$$

$\ldots\ldots\ldots(4)$

**Scheme 2: Derivation has been taken from Tailor series.**

$$\frac{\partial^2 P}{\partial r^2} = \frac{1}{3(\Delta r)^2}\begin{bmatrix} P_{(i+1,j+1)} - 2P_{(i,j+1)} + P_{(i-1,j+1)} + P_{(i+1,j)} - 2P_{(i,j)} \\ +P_{(i-1,j)} + P_{(i+1,j-1)} - 2P_{(i,j-1)} + P_{(i-1,j-1)} \end{bmatrix}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

$$\frac{\partial^2 P}{\partial \theta^2} = \frac{1}{3(\Delta \theta)^2}\begin{bmatrix} P_{(i+1,j+1)} - 2P_{(i+1,j)} + P_{(i+1,j-1)} + P_{(i,j+1)} - 2P_{(i,j)} \\ +P_{(i,j-1)} + P_{(i-1,j+1)} - 2P_{(i-1,j)} + P_{(i-1,j-1)} \end{bmatrix}\ldots\ldots\ldots\ldots\ldots(6)$$

$\alpha = \left[\dfrac{3r^2.\Delta r^2.\Delta\theta^2}{2(\Delta r^2 + \Delta\theta^2.r^2)}\right]$ is a weighting Factor

$$\nabla^2\omega_{(i,j)} = [\alpha].\left[\begin{array}{l} \dfrac{1}{3(\Delta r)^2}\left[P_{(i+1,j+1)} - 2P_{(i,j+1)} + P_{(i-1,j+1)} + P_{(i+1,j)} - 2P_{(i,j)} + P_{(i-1,j)} + P_{(i+1,j-1)} - 2P_{(i,j-1)} + 2P_{(i-1,j-1)}\right] + \\ \dfrac{1}{r}\dfrac{1}{6(\Delta r)}\left[P_{(i+1,j+1)} + P_{(i+1,j-1)} - P_{(i-1,j+1)} - P_{(i-1,j-1)} + P_{(i+1,j)} - P_{(i-1,j)}\right] + \\ \dfrac{1}{r^2}\dfrac{1}{3(\Delta\theta)^2}\left[P_{(i+1,j+1)} - 2P_{(i,+1j)} + P_{(i+1,j-1)} + P_{(i,j+1)} - 2P_{(i,j)} + P_{(i,j-1)} + P_{(i-1,j+1)} - 2P_{(i-1,j)} + P_{(i-1,j-1)}\right] \end{array}\right] - P_{(i,j)}$$

**Results with Example:**

For the discretization scheme following notations are adopted. $r_a = 3$ Fixed in computation

$p_{i,j} = r^3 sin\theta \;\; and\; r_a = i(\Delta r) \;\; i = 0,1,2,3 \dots \dots . n$ And
$\theta_j = j(\Delta\theta) \qquad j = 0,1,2,3,\dots,m$

In the polar mesh system following values of the constants are chosen $\Delta r = 0.1$, $\Delta\theta = \pi/180 radians$

| | | Result of Scheme 1 | | | Result of Scheme 2 | | |
|---|---|---|---|---|---|---|---|
| **r** | **Radian** | **Exact Value** | **Approximate value** | **Error** | **Exact Value** | **Approximate value** | **Error** |
| 3.1 | 0.017453293 | 0.009074857 | 0.009089024 | 1.42E-05 | 0.009074857 | 0.009089024 | -1.42E-05 |
| 3.1 | 0.034906585 | 0.01814971 | 0.018178048 | -2.83E-05 | 0.018149711 | 0.018178048 | -2.83E-05 |
| 3.1 | 0.052359878 | 0.05235988 | 0.027267067 | -4.25E-05 | 0.052359878 | 0.027224567 | -4.25E-05 |
| 3.1 | 0.06981317 | 0.03629941 | 0.036356088 | -5.67E-05 | 0.06981317 | 0.036299415 | -5.67E-05 |
| 3.1 | 0.08726646 | 0.04537427 | 0.04544510 | -7.08E- | 0.08726646 | 0.04537426 | -7.08E-05 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 3 | | 3 | 05 | 3 | 7 | |
| 3.2 | 0.017453293 | 0.0099817 | 0.009996327 | -1.46E-05 | 0.017453293 | 0.009981704 | -1.46E-05 |
| 3.2 | 0.034906585 | 0.01996341 | 0.019992653 | -2.92E-05 | 0.034906585 | 0.019963406 | -2.92E-05 |
| 3.2 | 0.052359878 | 0.02994511 | 0.029988976 | -4.39E-05 | 0.052359878 | 0.029945109 | -4.39E-05 |
| 3.2 | 0.06981317 | 0.03992681 | 0.039985303 | -5.85E-05 | 0.06981317 | 0.039926808 | -5.85E-05 |
| 3.2 | 0.087266463 | 0.0499085 | 0.04998162 | -7.31E-05 | 0.087266463 | 0.049908504 | -7.31E-05 |
| 3.3 | 0.017453293 | 0.01094704 | 0.010962118 | -1.51E-05 | 0.017453293 | 0.010947037 | -1.51E-05 |
| 3.3 | 0.034906585 | 0.02189441 | 0.021924233 | -3.02E-05 | 0.034906585 | 0.021894071 | -3.02E-05 |
| 3.3 | 0.052359878 | 0.03284111 | 0.032886349 | -4.52E-05 | 0.032841105 | 0.032886349 | -4.5244E-05 |
| 3.3 | 0.06981317 | 0.04378814 | 0.043848459 | -6.03E-05 | 0.043788135 | 0.043848459 | -6.0324E-05 |
| 3.3 | 0.087266463 | 0.05473516 | 0.054810569 | -7.54E-05 | 0.054735161 | 0.054810569 | -7.5407E-05 |

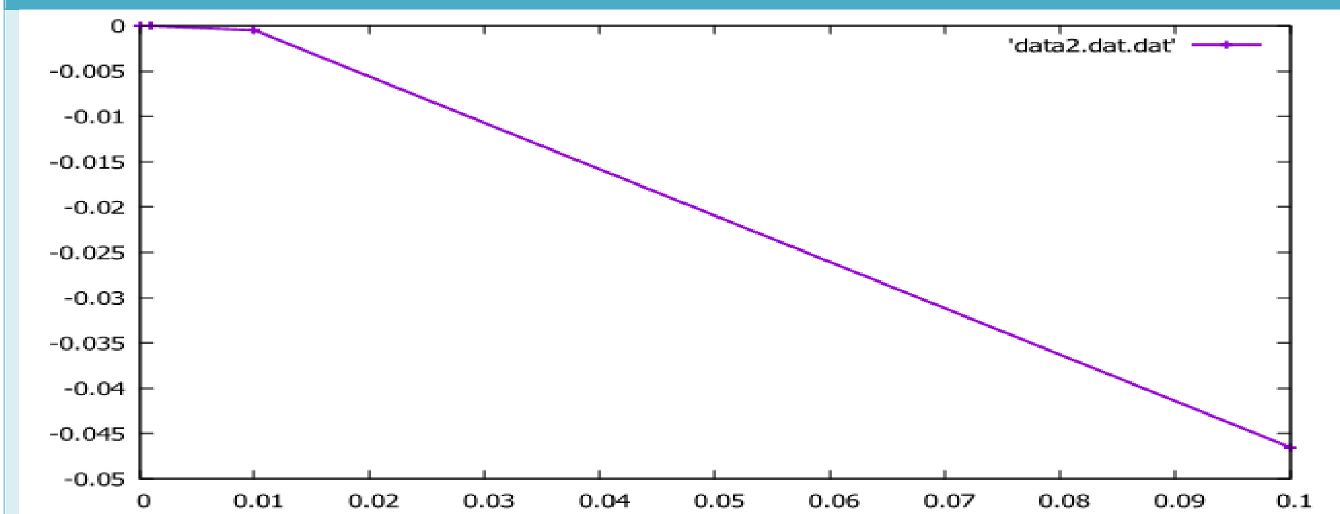
**Results of Developed Scheme**

**ERROR ANALYSIS OF SCHEMES.**

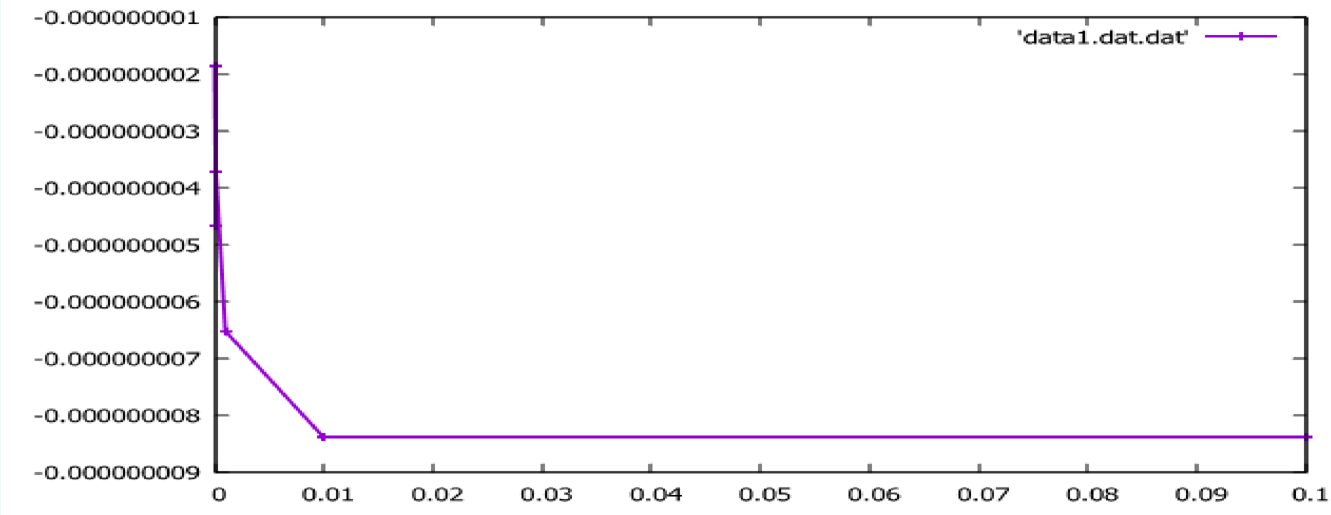
| Error of Scheme 1 | | Error of Scheme 2 | |
|---|---|---|---|
| $Change\ in\ \Delta r$ | $ChangeinError$ | $Changein\ \Delta r$ | $ChangeinError$ |
| **0.000001** | **-5.7220459E-06** | **0.000001** | **-1.86264515E-09** |
| **0.00001** | **0** | **0.00001** | **-1.86264515E-09** |
| **0.0001** | **-3.8146973E-06** | **0.0001** | **-3.72529030E-09** |
| **0.001** | **-1.1440918E-05** | **0.001** | **-6.51922258E-09** |
| **0.01** | **-4.7492981E-04** | **0.01** | **-8.38190032E-09** |
| **0.1** | **-4.6508789E-02** | **0.1** | **-8.38190032E-09** |

**Error Analysis by Graphically**

**Error of Scheme 1**



**Error of Scheme 2**




**Stability Results of Proposed Schemes**

**Scheme 1:**

The estimation consequences have achieved by plummeting Δr beside error. This is taken from the suggested structure of given molecule in the discretization of Polar coordinate system. Here for observation we have reserved some changed standards of Δr which easily visualize in above figure 1 and the indicated figure shows the error decreases linearly and quickly among the standards of Δr 0.1 near to 0.00998302.So the decreasing value from 0.00998302, error converge very fast to 0.

**Scheme 2:**

The scientific outcomes has been found out with the help of CDS method which is being applied on discretization of laplacian operator on nine point stencil on PCS. These results have found out by reducing Δr beside error. Here the pictorial form of selected values of Δr and his error specification can be seen in figure 2 and in this figure you can easily analyzed the standards of outcomes where the error is falling lineally and hastily among the value of Δr (0.01 to 0.000966277). The convergence criteria of this scheme is very quickly to be zero because of his decreasing value Δr up to 0.00966277. The scheme 2 shows the poor stable scheme due to his more fast convergence to zero in the discretization of laplacian operator.

**Review on Consequences of Schemes.**

**Scheme 1:**

The error of this scheme is find out by the investigation on discretization of laplacian operator which is in contradiction of several reduces values of $\Delta r$ and his scheme's outcomes can be visualized in table 1 and pictorial data is mentioned in figure 1.Here consequences indicates that whenever the error approaches 0 as $\Delta r$ approaches 0. This can be exposed and this is being established on computational scheme of discretization of laplacian operator which is adequately unwavering.

**Scheme 2:**

In this scheme same kind of strategies have applied for observation of planned proposed scheme which is discretised on nine point stencil on polar coordinate system. In this scheme by the investigation, error consequences have acquired against the different outcomes of $\Delta r$ which in shifted in table 2 and his molecule data can be observed in figure 2. In the results indicates that $\Delta r$ will be near about zero if error will approaches zero. So for the same guidelines and applied step we can say that the crested proposed computational scheme of discretization of laplacian equation is appropriately steady and accurate.

**Discussion on the Schemes**

**Scheme 1:**

The consequences of this scheme, which is being discretized on 9 point stencil on PCS has been gained by implementing the CDS and FORTRAN language code with addition of open Dx Software.All the outcomes of discretization are developed and listed in table 1. The computational outcomes shows the accuracy and stability of scheme 1 and in these outcomes of isotropic isolated laplacian operator, the accuracy and stability will be counted more than 6 fraction points.

**Scheme 2:**

In this planned scheme 2, which is discretized on nine point stencil of laplacian operator in PCS. Here we have developed another CDS and FORTRAN language code to development of the some computational, scientific and graphical consequences of scheme 2. After implementing the code, the gained results of planned scheme has compared with the analytic consequences of proposed scheme 2 for checking the error analysis and stability purposes. Which can be visualized in table 2.The computational outcomes of this discretized laplacian operator displays suitable precision. In the above observation of the discretization of proposed scheme 2 the precisionvestiges more than nine unit points and the error is lesser the first scheme.

## Future Work

i. For the evaluation of discretization of laplacian operator in PCS on 13,17 & 25 stencils, the approximation of Finite difference method will be applied including PDES with mixed derivatives.

ii. Work on the Literature of pdes and mixed pdes is needed by implementing the finite difference method on laplacian operator, on not only cylindrical but spherical coordinate system on 13,17 & 25 stencils as well as.

## Examples

i. Solve $U_{r,\theta} = r^3 cos\theta$.

ii. Solve $U_{r,\theta} = r^3 tan\theta$.

## References

[1] A. Bruno-alfonso and H. A. Navarro, "Alternate treatments of jacobian singularities in polar coordinates within," vol. 8, no. 3, pp. 163–171, 2012.

[2] A. S. Reimer and A. F. Cheviakov, "A Matlab-based finite-difference solver for the Poisson problem with mixed Dirichlet-Neumann boundary conditions," *Comput. Phys. Commun.*, vol. 184, no. 3, pp. 783–798, 2013.

[3] A. K. Mitra, "Finite Difference Method for the Solution of Laplace Equation."

[4] E. Abstract and Y. Wang, "Discrete Laplace Operator on Meshed Surfaces."

[5] M. Esmaeilzadeh, "Scholarship at UWindsor A Cartesian Cut - Stencil Method for the Finite Difference Solution of PDEs in Complex Domains," 2016.

[6] S. L. M. El-hachem and J. T. M. Reggio, "High Order Spatial Generalization of 2D and 3D Isotropic Discrete Gradient Operators with Fast Evaluation on GPUs," pp. 545–573, 2014.

7] M. Lai, Z. Li, and X. Lin, "Fast solvers for 3D Poisson equations involving interfaces in a finite or the infinite domain," vol. 191, pp. 106–125, 2006.

[8] E. Letters, T. Madras, and M. Sciences, “Lattice differential operators for computational physics,” no. January 2017, 2013.

[9] V. Thomã, “From ÿnite di erences to ÿnite elements A short history of numerical analysis of partial di erential equations,” vol. 128, pp. 1–54, 2001.

[10] A. Kumar, “Isotropic finite-differences,” *J. Comput. Phys.*, vol. 201, no. 1, pp. 109–118, 2004.

[11] C. Shin and H. Sohn, “A frequency-space 2-D scalar wave extrapolator using extended 25-point finite-difference operator,” *Geophysics*, vol. 63, no. 1, pp. 289–296, 1998.

[12] F. Costa, D. De Oliveira, E. Ogasawara, A. A. B. Lima, and M. Mattoso, “Resource.