SOLVING A CLASS OF TWO-DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATIONS AS A GENERALIZED PROBLEM OF MOMENTS

[12]María B. Pintarelli

[1]*Departamento de Matematica de la Facultad de Ciencias Exactas*
Universidad Nacional de La Plata, LaPlata -1900. Argentina

[2]*Departmento de Ciencias Basicas de la Facultad de Ingenieria*
Universidad Nacional de La Plata -1900. Argentina
Email: mariabpintarelli@gmail.com

Abstract

It will be shown that solve an equation two-dimensional Volterra nonlinear can be solved numerically applying the techniques of inverse generalized moments problem in two steps writing the Volterra's equation as a Klein-Gordon equation of the form $w_{tt} - w_{xx} = H(x,t)$, which $H(x,t)$ it is unknown. In a first step, $H(x,t)$ is numerically approximate, and in a second step we numerically approximate the solution of Klein-Gordon equation using the $H(x,t)$ previously approximated.The method is illustrated with examples.

Keywords: *Klein-Gordon, nonlinear Volterra integral equations, generalized moment problem, inverse problem*.

INTRODUCTION

We want to find $w(x,t)$ such that

$$w(x,t) - \int_0^t \int_0^x K(x,t,y,z,w(y,z))dydz = f(x,t) \quad (x,y) \epsilon D$$

Where $w(x,t)$ is unknown function and the functions $f(x,t)$ known continuous about a D domain where $D = \{(x,t); x > 0, t > 0\}$ and $K(x,t,y,z,w)$ known continuous about a E domain where $E = \{(x,t,y,z); 0 \leq y \leq x,\ 0 \leq z \leq t,\ x > 0, t > 0, -\infty < w < \infty\}$.

Also f and K are r times continuously differentiable over D and E respectively with r = 2. In that case the solution w will be r = 2 times continuously differentiable over D. The underlying space is $L^2(D)$

Note that

$$w(0,t) = f(0,t) \quad , \quad w(x,0) = f(x,0) \qquad t \geq 0 \ , \quad x \geq 0$$

$$w_x(0,t) \quad , \quad w_t(x,0) \quad t \geq 0 \ , \quad x \geq 0$$ they are known

Integral equations is a special topic in Applied Mathematics, as they constitute an important tool to model many problems in fields such as engineering, astrophysics, chemistry, quantum mechanics and many other fields. They are also applied in initial condition and boundary value problems for partial differential equations.

With so many applications, integral equations have been extensively studied. For example in [1] is investigated a collocation method for the approximate solution of Hammerstein integral equations in two dimensions. In [2] a numerical technique based on the Sinc collocation method is presented for the solution of two-dimensional Volterra integral equations of first and second kinds. The Sinc function properties are provided and the global convergence analysis is obtained to guarantee the efficiently of our method. In [3] a class of two-dimensional linear and nonlinear Volterra integral equations is solved by means of an analytic technique, namely the Homotopy analysis method (HAM). In [4] a numerical iterative algorithm based on combination of the successive approximations method and the quadrature formula for solving two-dimensional nonlinear Volterra integral equations is proposed. This algorithm uses a trapezoidal quadrature rule for Lipschitzian functions applied at each iterative step.

In paper [5], we develop two-step collocation (2-SC) methods to solve two-dimensional nonlinear Volterra integral equations (2D-NVIEs) of the second kind. Here we convert a 2D-NVIE of the second kind to a one-dimensional case, and then we solve the resulting equation numerically by two-step collocation methods.

In [6] the approximate solutions for two different type of two-dimensional nonlinear integral equations: two-dimensional nonlinear Volterra-Fredholm integral equations and the nonlinear mixed Volterra-Fredholm integral equations are obtained using the Laguerre wavelet method. To do this, these two-dimensional nonlinear integral equations are transformed into a system of nonlinear algebraic equations in matrix form.

In [7] se proponen new theorems of the reduced differential transform method (RDTM) for solving a class of two-dimensional linear and nonlinear Volterra integral equations (VIEs) of the second kind.

In [8] the rational Haar wavelet method has been used to solve the two-dimensional Volterra integral equations. Numerical solutions and the rate of convergence, are presented.

In [9] Using fixed-point techniques and Faber Schauder systems in Banach spaces, is obtained an approximation of the solution of two-dimensional nonlinear Volterra, Fredholm and mixed Volterra-Fredholm integral equations.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation.

The interest is not to compare with the existing methods, but to present a different method to my novel criteria.

The generalized moments problem [10,11,12], is to find a function $f(x)$ about a domain $\Omega \subset R^d$ that satisfies the sequence of equations

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i \epsilon N ----------- \quad (1)$$

where N is the set of the natural numbers, $(g_i(x))$ is a given sequence of functions in $L^2(\Omega)$ linearly independent known and the succession of real numbers $\{\mu_i\}_{i \epsilon N}$ are known data.

The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [10,11,12]. There are several methods to build regularized solutions. One of them is the truncated expansion method [10].

This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i = 1,2,\dots,n. -------- \quad (2)$$

where it is considered as approximate solution of to $p_n(x) = \sum_{i=0}^{n} \lambda_i \phi_i(x)$, and the functions $\{\phi_i(x)\}_{i=1,..,n}$ result of orthonormalize $\langle g_1, g_2, \dots, g_n \rangle$ being λ_i the coefficients based on the data μ_i. In the subspace generated by $\langle g_1, g_2, \dots, g_n \rangle$ the solution is stable. If $n \epsilon N$ is chosen in an appropriate way then the solution of (2) it approaches the solution of the problem (1).

In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution (pages 19 - 30 of [10]).

ARTICLE ORGANIZATION

To find $w(x,t)$

$$w(x,t) - \int_0^t \int_0^x K(x,t,y,z,w(y,z)) dy dz = f(x,t) \quad (x,y) \epsilon D$$

Where $w(x,t)$ is unknown function and the function $f(x,t)$ known continuous about a D domain where $D = \{(x,t); x > 0, t > 0\}$ and $K(x,t,y,z,w)$ known continuous about a E

domain where $E = \{(x,t,y,z); 0 \le y \le x,\ 0 \le z \le t\ ,\ x > 0\ , t > 0, -\infty < w < \infty\}$

$$w(0,t) = f(0,t) \quad , \quad w(x,0) = f(x,0) \qquad t \ge 0 \quad x \ge 0$$

$w_x(0,t) \quad , \quad w_t(x,0) \quad t \ge 0 \qquad x \ge 0$ they are known

We will do it in two steps. The next section describes the first step. The section that follows explains the second step. Then it is explained how the generalized moment problem is solved with the truncated expansion method. Finally the numerical example and the conclusions.

FIRST STEP

We consider

$$w(x,t) - \int_0^t \int_0^x K(x,t,y,z,w(y,z))dydz = f(x,t) \quad ------ \quad (3)$$

We differentiate (3) with respect to *t* twice:

$$w_{tt}(x,t) = \left(\int_0^t \int_0^x K(x,t,y,z,w(y,z))dydz\right)_{tt} + f_{tt}(x,t) = R(x,t)$$

$$w_{tt}(x,t) - w_{xx}(x,t) = R(x,t) - w_{xx}(x,t) = H(x,t)$$

the conditions are:

$$w(0,t) = f(0,t) \quad , \quad w(x,0) = f(x,0) \qquad t \ge 0 \quad x \ge 0$$

Note that

$$w_t(x,t) = \int_0^t \int_0^x K_t(x,t,y,z,w(y,z))dydz + \int_0^x K(x,t,y,t,w(y,t))dy + f_t\,(x,t)$$

Therefore

$$w_t(x,0) = \int_0^x K(x\,,0,y,0,w(y,0))dy + f_t(x,0)$$

Analogously

$$w_x(0,t) = \int_0^t K(0\,,t,0,z,w(0,t))dz + f_x(0,t)$$

That is $w_x(0,t) \quad t \geq 0$, $w_t(x,0) \quad \boldsymbol{x} \geq 0$ they are known

We take as an auxiliary function

$$u(m,r,x,t) = e^{-mx}e^{-rt}$$

Note that $u_{xx} = m^2u$ and $u_{tt} = r^2u$.

We consider

$$w_{xx}(x,t) - w_{tt}(x,t) = -H(x,t)$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w_t)$$

Since $div(F^*) = -H(x,t)$ we have to:

$$\iint_D u\, div(F^*)dA = \iint_D u\left(-H(x,t)\right)dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so

$$\iint_D u\, div(F^*)dA = \iint_D div(uF^*)dA - \iint_D F^*.\nabla u dA$$

Where $\nabla u = (u_x, u_t)$.

And

$$\iint_D F^* . \nabla u dA = \iint_D (F_1 u_x + F_2 u_t) dA$$

Integrating by parts with respect to x:

$$\iint_D F_1 \, u_x dA = \int_0^\infty \int_0^\infty F_1 \, u_x dx dt =$$

$$= \int_0^\infty \big(-w(a_1, t) u_x(m, r, a_1, t)\big) dt - \iint_D w u_{xx} \, dA =$$

$$\int_0^\infty \big(-w(0, t) u_x(m, r, 0, t)\big) dt - \iint_D w \, (m)^2 u dA$$

Analogously

$$\iint_D F_2 \, u_t dA = \int_0^\infty \int_0^\infty F_2 \, u_t dx dt = \int_0^\infty \big(-w(x, 0) u_t(m, r, x, 0)\big) dt - \int_0^\infty \int_0^\infty w(r)^2 u dx dt$$

then

$$\iint_D F^* . \nabla u dA = \int_0^\infty \big(-w(0, t) u_x(m, r, 0, t)\big) dt -$$

$$-\int_0^\infty \big(-w(x, 0) u_t(m, r, x, 0)\big) dt - \iint_D w \, u(m^2 - r^2) dA = A(m, r)$$

On the other hand,

$$\int_C (uF^*) . nds =$$

$$= \int_0^\infty u(m, r, x, 0) w_t(x, 0) dx - \int_0^\infty u(m, r, 0, t) w_x(0, t) dt = G(m, r)$$

$$\therefore \iint u \big(-H(x, t)\big) dA = G(m, r) - A(m, r) + \iint_D w \, u(m^2 - r^2) dA$$

So if we do $r = m$:

$$\iint u\left(-H(x,t)\right)dA = G(m,m) - A(m,m)$$

That is

$$\iint u\left(H(x,t)\right)dA = -G(m,m) + A(m,m) = \phi(m)$$

To solve this integral equation we give integer values to m, $m = 0,1,2, \dots n$

Then

$$\int_{a_1}^{\infty} H(x,t) R_m(x,t) dx = \phi(m) = \mu_m - - - - - - - - (4)$$

We interpret (4) as a generalized moments problem.

$p_{1n}(x,t)$ is the numerical approximation found with the truncated expansion method for $H(x,t)$, with $R_m(x,t) = u(m,m,x,t) = e^{-mx}\, e^{-mt}$ $m = 0,1,2, \dots n$ where n is conveniently chosen.

In section 4 the truncated expansion method will be explained in detail and a theorem will be given that explains what is the accuracy of the approximation found by this method.

APPROACH TO $w(x,t)$ - SECOND STEP

To find an approximation of $w(x,t)$ a similar approach to the previous one is made where $H(x,t)$ is replaced by $p_{1n}(x)$ and we do not consider $r = m$.

We take the auxiliary function $u(m,r,x,t) = e^{-(m+1)x} e^{-(r+1)t}$.

Note that $u_{xx} = (m+1)^2 u$ and $u_{tt} = (r+1)^2 u$.

We define the vector field $F^* = \left(F_1(w), F_2(w)\right) = (w_x, -w_t)$

Since $div(F^*) = -H(x,t)$ we have to:

$$\iint_D u\, div(F^*) dA = \iint_D u\left(-H(x,t)\right)dA$$

In addition, as $u div(F^*) = div(uF^*) - F^*.\nabla u$, so:

$$\iint_D u\, div(F^*) dA = \iint_D div(uF^*) dA - \iint_D F^*.\nabla u dA$$

Thus

$$\therefore \iint u\left(-H(x,t)\right)dA = G(m,r) - A(m,r) + \iint_D w\,u((m+1)^2 - (r+1)^2)dA$$

Then

$$\iint_D w\,u((m+1)^2 - (r+1)^2)dA = -G(m,r) + A(m,r) + \iint_D u\,H(x,t)dA$$

where $G(m,r)\ y\ A(m,r)$ they are like before.

We replace $H(x,t)$ by $p_{1n}(x)$ and then

$$\iint_D w(x,t)H_{mr}(x,t)dA = \frac{-G(m,r) + A(m,r) + \iint_D u\,p_{1n}(x)\,dA}{((m+1)^2 - (r+1)^2)} = \phi(m,r)$$
$$= \mu_{mr} - - - - (5)$$

where

$$H_{m,r}(x) = u(m,r,x,t)$$

We can consider (5) as a two-dimensional generalized moment problem if we discretize giving m and r non-negative integer values $m = 0,1,2,\ldots n_1$; $r = 0,1,2,\ldots,n_2$, where n_1 and n_2 are conveniently chosen.

An approximation $p_{2n}(x,t)$ is found by the truncated expansion method for $w(x,t)$ where $n = n_1 \times n_2$.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [12] and generalized in [13] and [14] to find an approximation $p_n(x,t)$ for the corresponding finite problem with $i = 0,1,2,\ldots,n$ where n is the number of moments μ_i. We consider the basis $\phi_i(x,t)$ $i = 0,1,2,\ldots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t)$ $i = 0,1,2,\ldots,n$.

We approximate the solution $\mathrm{w(x,t)}$ with [12] and generalized in [13] y [14]:

$$p_n(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t)$$

where

$$\lambda_i = \sum_{j=0}^{i} C_{ij}\,\mu_j \quad i = 0,1,2,\dots,n$$

And the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1}(-1)\frac{\langle H_i(x,t)|\phi_k(x,t)\rangle}{\|\phi_k(x,t)\|^2}C_{kj}\right).\|\phi_i(x,t)\|^{-1} \quad 1 < i \le n\,;1 \le j < i\,.$$

The terms of the diagonal are $\|\phi_i(x,t)\|^{-1}$ $i = 0,1,\dots,n$.

The proof of the following theorem is in [14,15].

In [15] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [16] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

Theorem

We considerer $b_1 = b_2 = \infty$.

Sea $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t)\,\epsilon\,L^2((a_1,b_1)\times(a_2,b_2))$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n}\left|\iint_E H_i\,(x,t)f(x,t)dtdx - \mu_i\right|^2 \le \varepsilon^2 \quad .$$

$$\iint_E \left(x\,{f_x}^2 + t\,{f_t}^2\right) Exp[x+t]dtdx \le M^2 - - - - - - - (6).$$

And it must be fulfilled that

$$t^i f(x,t) \to 0 \quad \text{si} \quad t\to\infty \quad \text{para todo} \quad i \in N$$

then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f(x,t)|^2\,dtdx \le min_i\left\{\|C^T C\|\varepsilon^2 + \frac{1}{8(n+1)^2}M^2; i = 0,1,\dots,n\right\}$$

where *C* it is a triangular matrix with elements C_{ij} $\quad (1 < i \le n;\ 1 \le j < i)$

and

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f(x,t) - p_n(x,t)|^2\,dtdx \le \|C^T C\|\varepsilon^2 + \frac{1}{8(n+1)^2}M^2.$$

If b_2 it is not infinite then (6) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}((b_1-a_1)^2 f_x^2 + (b_2-a_2)^2 f_t^2)dx\,dt \le M^2$$

NUMERICAL EXAMPLES

Example 1

We consider the equation

$w(x,t) - \int_0^t\int_0^x e^{-(x+y)}\sqrt{w(y,z)}dydz = e^{-t-\frac{x^2}{20}} - 2e^{10-\frac{t}{2}-x}(-1+e^{\frac{t}{2}})\sqrt{10\pi}\,(Erf(\sqrt{10}) + Erf(\frac{-20+x}{2\sqrt{10}}))$

whose solution is: $w(x,t) = e^{-t-\frac{x^2}{20}}$

we take n=5 moments and is approaching $p_{1n}(x,t) \approx H(x,t)$ where the accuracy is

$$\int_0^\infty\int_0^\infty e^{-x-t}\,(p_{1n}(x,t) - H(x,t))^2 dtdx = 0.170473$$

We consider this norm since the basis is $e^{-mx}\ e^{-mt} \quad m = 1,2, \ldots 5$.

In the Fig.1 the graphics of: $p_{15}(x,t)$ (magenta color) and $H(x,t)$ (light blue color) are superimposed.

we take $n = 6$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^\infty \int_0^\infty e^{-2x-3t}\left(p_{26}(x,t) - w(x,t)\right)^2 dtdx = 0.00899724$$

We consider this norm since the basis is $\{e^{-3t-2x}, e^{-4t-2x}, e^{-4t-3x}, e^{-5t-3x}, e^{-5t-4x}, e^{-6t-4x}\}$.

In the Fig. 2 the graphics of:$p_{26}(x)$(magenta color) and $w(x,t)$ (light blue color) are superimposed.

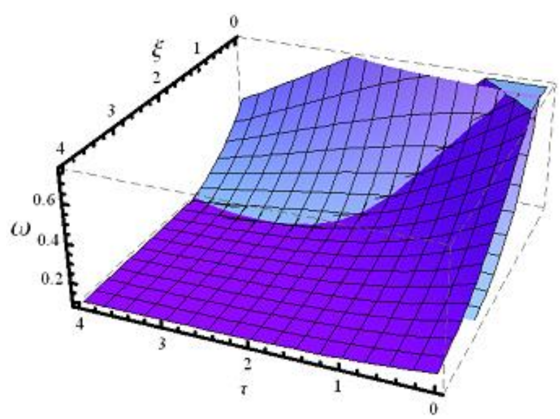

Fig. 1: $p_{15}(x,t)$ and $H(x,t)$ example 1

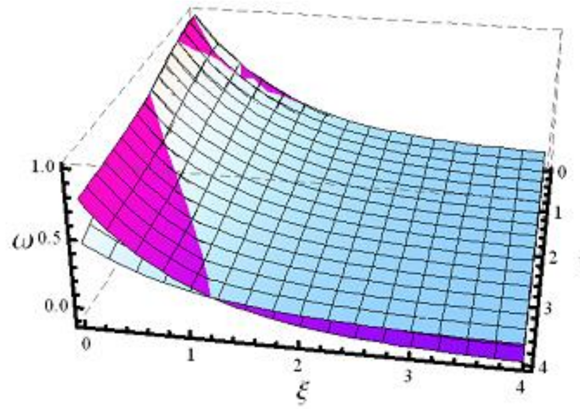

Fig. 2: $w(x,t)$ and $p_{26}(x,t)$ example 1

Example 2

We consider the equation

$$w(x,t) - \int_0^t \int_0^x (sin(x)+2)e^{-(x-y)}\sqrt{w(y,z)}dydz = \frac{e^{-x-t}}{1+x}$$
$$-2\sqrt{2}\left[-2e^{-x}(1-e^{-\frac{t}{2}})DawsonF\left(\frac{1}{\sqrt{2}}\right)\right.$$
$$\left.-2\sqrt{1+x}\left(\sqrt{\frac{e^{-x-t}}{1+x}}-\sqrt{\frac{e^{-x}}{1+x}}\right)DawsonF\left(\sqrt{\frac{1+x}{2}}\right)\right](2+sin(x))$$

whose solution is:$w(x,t) = \frac{e^{-(t+x)}}{1+x}$

we take n=5 moments and is approaching $p_{1n}(x,t) \approx H(x,t)$ where the accuracy is

$$\int_0^\infty \int_0^\infty e^{-x-t}\,(p_{1n}(x,t) - H(x,t))^2 dtdx = 0.237494$$

We consider this norm since the basis is $e^{-mx}\,e^{-mt} \quad m = 1,2,\ldots 5$.

In the Fig.1 the graphics of: $p_{15}(x,t)$ (magenta color) and $H(x,t)$ (light blue color) are superimposed.

we take $n = 6$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^\infty \int_0^\infty e^{-2x-3t}\big(p_{26}(x,t) - w(x,t)\big)^2\, dtdx = 0.00838925$$

We consider this norm since the basis is $\{e^{-3t-2x}, e^{-4t-2x}, e^{-4t-3x}, e^{-5t-3x}, e^{-5t-4x}, e^{-6t-4x}\}$.

In the Fig. 4 the graphics of:$p_{26}(x)$(magenta color) and $\varphi(x)$ (light blue color) are superimposed.

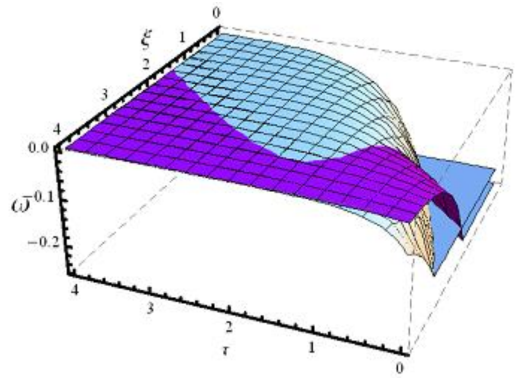

Fig. 3: $p_{15}(x)$ and $H(x,t)$ example 2

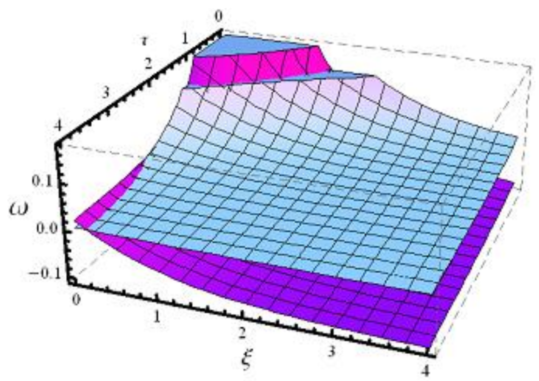

Fig. 4: $w(x,t)$ and $p_{26}(x,t)$ example 2

CONCLUSION

To find $w(x,t)$

$$w(x,t) - \int_0^t \int_0^x K(x,t,y,z,w(y,z))dydz = f(x,t) \quad (x,y) \epsilon D$$

where $w(x,t)$is unknown function and the functions $f(x,t)$ and $K(x,t,y,z,w)$ known continuous about a D domain and E domain respectively with $D = \{(x,t); x > 0, t > 0\}$ and $E = \{(x,t,y,z); 0 \le y \le x, \ 0 \le z \le t, \ x > 0, t > 0, -\infty < w < \infty\}$ we will do it in two steps.

We differentiate with respect to *t* twice and consider the equation in partial derivatives of second order

$$w_{xx}(x,t) - w_{tt}(x,t) = -H(x,t)$$

with $H(x,t)$ unknown.

1. In a first step we approximate $H(x,t)$ with $p_{1n}(x)$ solving the integral equation

$$\int_{a_1}^{\infty} H(x,t) R_m(x,t) dx = \phi(m) = \mu_m$$

which we interpret as a generalized moments problem and $R_m(x,t) = u(m,m,x,t) = e^{-mx}\, e^{-mt}$ $m = 0,1,2, \dots n$ where n is conveniently chosen.

2. To find an approximation of $w(x,t)$ we consider:

$$\iint_D w(x,t) H_{mr}(x,t) dA = \frac{-G(m,r) + A(m,r) + \iint_D u\, p_{1n}(x) dA}{((m+1)^2 - (r+1)^2)} = \phi(m,r) = \mu_{mr}$$

where $H_{m,r}(x) = u(m,r,x,t)$. We can consider it as a two-dimensional generalized moment problem if we discretize giving m and r non-negative integer values $m \; and \; r$. An approximation $p_{2n}(x,t)$ is found by the truncated expansion method for $w(x,t)$ where $n = n_1 \times n_2$.

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