INVERSE SOURCE IDENTIFICATION ON POISSON EQUATION WITH CAUCHY CONDITIONS AS A GENERALIZED PROBLEM OF MOMENTS

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Abstract

We consider the problem of finding a pair of functions $\Phi(x,t)$and $w(x,t)$that satisfy the equation

$w_{xx}(x,t) + w_{tt}(x,t) = \Phi(x,t)$ under Cauchy boundary conditions. We will see that an approximate solution can be found using the techniques of generalized inverse problem of moments and find dimensions for the error of the estimated solution.

Keywords*: generalized moment problem; integral equations; Poisson equation, inverse source.*

INTRODUCTION

We want to find $\Phi(x,t)$and $w(x,t)$such that

$$\mathrm{w_{tt}}(x,t) + w_{xx}(x,t) = \Phi(\mathrm{x,t})$$

about a domain about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $\mathrm{E} = (\mathrm{a}_1, \mathrm{b}_1) \times (\mathrm{a}_2, \infty)$

under the Cauchy conditions using the problem generalized moments techniques. Unfortunately the source term $\Phi(x,t)$ is not always known. In most practical problems, this source is unattainable except for some of its harmonic components. In real-life applications some scattered measurements of data on the boundary are needed for the recovery of the unknown source term. This is called inverse source identification problem which is a typical ill-posed problem in the sense of

Hadamard [1]. In other words, any small error in the scattered measurement data may induce enormous error to the solution.

Inverse source identification problems arise from many branches of engineering disciplines. For instances, crack determination [2], [3], heat source determination [4], inverse heat conduction [5] electromagnetic source identification [6] and Stefan design [7] problems. Theoretical investigation on the inverse source identification problems can be found in the works of [8].

There are numerous works on the recovery of the source function in a Poisson equation for example [9,10, 11, 12] to name a few.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. The interest is not to compare with the existing methods, but to present a different method.

The generalized moments problem [13, 14, 15, 16, 17] is to find a function $f(x)$about a domain $\Omega \subset \mathrm{R}^{\mathrm{d}}$ that satisfies the sequence of equations

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i \epsilon N ------------ (1)$$

where N is the set of the natural numbers, $(\mathrm{g_i(x)})$ is a given sequence of functions in $\mathrm{L}^2(\Omega)$ linearly independent known and the succession of real numbers μ_i are known data.

The problem of Hausdorff moments [15, 17], is to find a function $f(x)$en (a, b) such that

$$\mu_i = \int_a^b x^i f(x) dx \quad i \epsilon N --------(2)$$

In this case $g_i(x) = x^i$ with *i* belonging to the set *N*.

If the integration interval is $(0, \infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty, \infty)$ we have the problem of Hamburger moments [15, 17].

The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [15, 16]. There are several methods to build regularized solutions. One of them is the truncated expansion method [16].

This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i = 1,2, \ldots, n - - - - - - - - - - (3)$$

where it is considered as approximate solution of $f(x)$ to

$$p_n(x) = \sum_{i=0}^{n} \lambda_i \, \phi_i(x)$$

and the functions $\phi_i(x)_{i=0,1,\ldots,n}$ result of orthonormalize $\langle g_1, g_2, \ldots, g_n \rangle$ being λ_i the coefficients based on the data μ_i . In the subspace generated by $\langle g_1, g_2, \ldots, g_n \rangle$ the solution is stable. If $n \in N$ is chosen in an appropriate way then the solution of (3) it approaches the solution of the original problem (1).

In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution ([16] p.p. 19 - 30).

ARTICLE ORGANIZATION

We want to find $\Phi(x,t)$and $w(x,t)$such that

$$w_{tt}(x,t) + w_{xx}(x,t) = \Phi(x,t)$$

about a domain about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$
under the Cauchy conditions using the problem generalized moments techniques.

We will see that the problem can be solved in three steps.
In steps one and two we found an approximation for $w(x,t)$. Then in a third step an approximate solution is found for $\Phi(x,t)$

The next section describes the first step.
The section that follows it is explained how the generalized moment problem is solved with the truncated expansion method and explains the second step where an approximation for $w(x,t)$ is found.
Finally the numerical examples and the conclusions.

APPROXIMATION OF $w(x,t)$.

You want to find $w(x,t)$ such that

$$w_{xx} + w_{tt} = \Phi(x,t) - - - - - - - - (4)$$

about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$
We consider

$$w_{xx} - k\, w_{tt} = -(k+1)\, w_{tt} + \Phi(x,t) = G(x,t) ------(5)$$

$If\ w_x \neq w_t$ we can take $k = 1$.

We take as an auxiliary function for the finite case

$$u(m,r,x,t) = e^{-m\left(\frac{x}{b1}\right)} e^{-r\left(\frac{t}{b2}\right)}$$

For the unbounded case

$$u(m,r,x,t) = e^{-m\left(\frac{x}{b1}\right)} e^{-r(t)}$$

In what follows we considerer the case with E bounded

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -kw_t)$$

As

$$udiv(F^*) = uG(x,t)$$

we have to:

$$\iint_E u\, div(F^*) dA = \iint_E u\, G(x,t) dA$$

Moreover, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, then

$$\iint_E u\, div(F^*) dA = \iint_E div(uF^*) dA - \iint_E F^*.\nabla u dA ---------- (6)$$

where $\nabla u = (u_x, u_t)$

Besides that

$$\iint_E div(uF^*) dA = \iint_E (uw_x)_x - (ukw_t)_t dA =$$

$$= \iint_E u\, div(F^*) dA + \iint_E (u_x w_x - u_t k w_t) dA ---------------- (7)$$

Then from (6) y (7):

$$\iint_E (u_x w_x - u_t k w_t) dA = \iint_E F^* . \nabla u dA - - - - - - - - - - - - - - (8)$$

On the other hand, it can be proven, after several calculations that, integrating by parts:

$$\iint_E F^* . \nabla u dA =$$

$$A(m,r) + B(m,r) - \iint_E u\, w \left(\left(\frac{m}{b_1}\right)^2 - k \left(\frac{r}{b_2}\right)^2 \right) dA = \varphi(m,r) - - - - - - - - - - (9)$$

with

$$A(m,r) = \int_{a_2}^{b_2} \left(-\frac{m}{b_1}\right) u(m,r,b_1,t) w(b_1,t) - \left(-\frac{m}{b_1}\right) u(m,r,a_1,t) w(a_1,t) dt$$

$$B(m,r) = \int_{a_1}^{b_1} \left(-\frac{r}{b_2}\right) u(m,r,x,b_2)(-k) w(x,b_2) - \left(-\frac{r}{b_2}\right) u(m,r,x,a_2)(-k) w(x,a_2) dx$$

If $\frac{m}{b_1} = \sqrt{k}\frac{r}{b_2}$, instead of (8) y (9):

$$\iint_E \left(-\sqrt{k}\frac{r}{b_2}\right) u\, w_x - \left(-\frac{r}{b_2}\right) k w_t u dA = \varphi\left(\sqrt{k} r \frac{b_1}{b_2}, r\right)$$

$$\therefore \iint_E u(-\sqrt{k} w_x + k w_t) dA = \frac{\varphi\left(\sqrt{k} r \frac{b_1}{b_2}, r\right)}{\frac{r}{b_2}}$$

with

$$\frac{\varphi\left(\sqrt{k} r \frac{b_1}{b_2}, r\right)}{\frac{r}{b_2}} =$$

$$= \int_{a_2}^{b_2} -\frac{\sqrt{k}}{b_2} r\, u\left(\sqrt{k} r \frac{b_1}{b_2}, r, b_1, t\right) w(b_1,t) + \frac{\sqrt{k}}{b_2} r u\left(\sqrt{k} r \frac{b_1}{b_2}, r, a_1, t\right) w(a_1,t) dt +$$

$$+ \int_{a_1}^{b_1} -u\left(\sqrt{k} r \frac{b_1}{b_2}, r, x, b_2\right)\left(-\frac{kr}{b_2}\right) w(x,b_2) - u\left(\sqrt{k} r \frac{b_1}{b_2}, r, x, a_2\right)\left(-\frac{kr}{b_2}\right) w(x,a_2) dx$$

We note $\varphi_1(r) = \frac{\varphi\left(\sqrt{k} r \frac{b_1}{b_2}, r\right)}{\frac{r}{b_2}}$, then

$$\iint_E u\left(-\sqrt{k}w_x + kw_t\right)dA = \varphi_1(r) ------------(10)$$

To solve this integral equation we take a base $\psi_i(r) = r^i e^{-r} \quad i = 0,1,2,\dots,n$ of $L^2(E)$.
Then we multiply both members of (10) by $\psi_i(r) = r^i e^{-r}$ and we integrate with respect to r, we obtain

$$\iint_E H_i(x,t)\left(-\sqrt{k}w_x + kw_t\right)dA = \int_{a_2}^{b_2} \varphi_1(r)\psi_i(r)dr = \mu_i \quad i = 0,1,2,\dots,n ----(11)$$

where $H_i(x,t) = \int_{a_2}^{b_2} u\left(-\sqrt{k}r\frac{b_1}{b_2}, r, x, t\right)\psi_i(r)dr$.
We can interpret (11) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation $p_{1n}(x,t)$ for $-\sqrt{k}w_x + kw_t$.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [17] and generalized in [19] and [20] to find an approximation $p_{1n}(x,t)$ of $-\sqrt{k}w_x + kw_t$ for the corresponding finite problem with $i = 0,1,2,\dots,n$, where n is the number of moments μ_i.
We consider the basis $\phi_i(x,t) \quad i = 0,1,2,\dots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t) \quad i = 0,1,2,\dots,n$.
We approximate the solution $-\sqrt{k}w_x + kw_t$ with [13] and generalized in [19] y [20]:

$$p_{1n}(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{where} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\,\mu_j \qquad i = 0,1,2,\dots,n$$

and the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1)\frac{\langle H_i(x,t)|\phi_k(x,t)\rangle}{\|\phi_k(x,t)\|^2} C_{kj}\right).\|\phi_i(x,t)\|^{-1} \quad 1 < i \le n\,; 1 \le j < i\,.$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0,1,\dots,n.$$

The proof of the following theorem is in [19] and [20].
In [20] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [18] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

Theorem

Let $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2\big((a_1,b_1)\times(a_2,b_2)\big)$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n}\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} H_i(x,t)f(x,t)dx\,dt-\mu_i\right|^2 \le \varepsilon^2$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left((b_1-a_1)^2 f_x^2+(b_2-a_2)^2 f_t^2\right)dx\,dt \le M^2 ------------ (12)$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f(x,t)|^2\,dtdx \le min_i\left\{\|C^TC\|\varepsilon^2+\frac{M^2}{8(n+1)^2};i=0,1,\dots,n\right\}$$

where C it is a triangular matrix with elements C_{ij}, $1 < i \le n;\ 1 \le j < i$
and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_{1n}(x,t)-f(x,t)|^2\,dxdt \le \|CC^T\|\varepsilon^2+\frac{M^2}{8(n+1)^2} ------------ (13)$$

If b_2 it is not finite then (12}) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left(xf_x^2+tf_t^2\right)dx\,dt \le M^2$$

And it must be fulfilled that $t^i f(x,t) \longrightarrow 0$ if $t \longrightarrow \infty$ $\forall i \epsilon N$.

So we have an equation in first order partial derivatives of the form

$$-\sqrt{k}w_x(x,t)+kw_t(x,t)=p_{1n}(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t)+A_2(x,t)w_t(x,t)=p_{1n}(x,t)$$

where $A_1(x,t)=-\sqrt{k}$ and $A_2(x,t)=k$.

It is resolved as in [18], that is, we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)w(x,t)dtdx = \varphi_2(m,r) - - - - - - - - (14)$$

With $K(m,r,x,t) = u(m,r,x,t)\left(\sqrt{k}\frac{m}{b_1} - k\frac{r}{b_2}\right)$ where now it is taken as an auxiliary function

$$u(m,r,x,t) = e^{-m\left(\frac{x}{b_1}\right)}e^{-r\left(\frac{t}{b_2}\right)}$$

and

$$\varphi_2(m,r) = \int_{a_1}^{b_1} u(m,r,x,b_2)kw(x,b_2) - u(m,r,x,a_2)kw(x,a_2)dx -$$
$$- \int_{a_2}^{b_2} u(m,r,b_1,t)\sqrt{k}\,w(b_1,t) - u(m,r,a_1,t)\sqrt{k}w(a_1,t)dt - \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_{1n}(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)} \qquad i = 0,1,\dots,n_1 \quad j = 0,1,2,\dots,n_2$$

and we multiply both members of (14) by $\psi_{ij}(m,r)$ and we integrate with respect to *m* and *r*

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(x,t)H_{ij}(x,t) = \mu_{ij} - - - - - - - - - - - (15)$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi_2(m,r)\psi_{ij}(m,r)dmdr$$

$$H_{ij}(x,t) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)\psi_{ij}(m,r)dmdr$$

We apply the truncated expansion method and find a numerical approximation $p_{2n}(x,t)$ for $w(x,t)$.

APPROXIMATION OF $\Phi(x,t)$

We consider

$$w_{xx} + w_{tt} = \Phi(x,t) ----------(16)$$

We take as an auxiliary function for the finite case

$$u(m,r,x,t) = cos\left(m\left(\frac{x}{b1}\right)\right)e^{-r\left(\frac{t}{b2}\right)}$$

For the unbounded case

$$u(m,r,x,t) = cos\left(m\left(\frac{x}{b1}\right)\right)e^{-r(t)}$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, w_t)$$

Since $div(F^*) = \Phi(x,t)$ we have to:

$$\iint_E u\, div(F^*)dA = \iint_E u\,\Phi(x,t)dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so

$$\iint_E u\, div(F^*)dA = \iint_E div(uF^*)dA - \iint_E F^*.\nabla u dA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_E F^*.\nabla u dA = \iint_E (F_1 u_x + F_2 u_t)dA$$

Integrating by parts:

$$\iint_E F_1\, u_x dA = \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_1\, u_x dxdt =$$

$$= \int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt - \iint_E wu_{xx}\, dA$$

Analogously

$$\iint_E F_2\, u_t dA = \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_2\, u_t dx dt =$$

$$= \int_{a_1}^{b_1}\big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx - \iint_E w u_{tt}\, dA$$

then

$$\iint_D F^*.\nabla u dA =$$

$$= \int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt +$$
$$+ \int_{a_1}^{b_1}\big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx -$$

$$-\iint_E w(u_{xx}+u_{tt})dA$$

where

$$\iint_E w(u_{xx}+u_{tt})dA = \iint_E wu\left(-\left(\frac{m}{b_1}\right)^2 + \left(\frac{r}{b_2}\right)^2\right)dA$$

On the other hand,

$$\int_C (uF^*).nds =$$

$$= \int_{a_1}^{b_1} u(m,r,x,b_2)w_t(x,a_2)dx - \int_{a_1}^{b_1} u(m,r,x,a_2)w_t(x,b_2)dx +$$

$$+ \int_{a_2}^{b_2} u(m,r,b_1,t)w_x(b_1,t)dt - \int_{a_2}^{b_2} u(m,r,a_1,t)w_x(a_1,t)dt = G(m,r)$$

$$\therefore \iint u\Phi(x,t)dA = G(m,r) -$$
$$-\int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt -$$

$$-\int_{a_1}^{b_1}\left(w(x,b_2)u_t(m,r,x,b_2)-w(x,a_2)u_t(m,r,x,a_2)\right)dx+\iint_E wu\left(-\left(\frac{m}{b_1}\right)^2+\left(\frac{r}{b_2}\right)^2\right)dA$$

We make $-\left(\frac{m}{b_1}\right)^2+\left(\frac{r}{b_2}\right)^2=0$, then (if $m\geq 0$, $r\geq 0$, $b_1>0, b_2>0$)

$$\iint u\left(m,\frac{m}{b_1}b_2,x,t\right)\Phi(x,t)dA=G\left(m,\frac{m}{b_1}b_2\right)-$$
$$-\int_{a_2}^{b_2}\left(w(b_1,t)u_x\left(m,\frac{m}{b_1}b_2,b_1,t\right)-w(a_1,t)u_x\left(m,\frac{m}{b_1}b_2,a_1,t\right)\right)dt-$$
$$-\int_{a_1}^{b_1}\left(w(x,b_2)u_t\left(m,\frac{m}{b_1}b_2,x,b_2\right)-w(x,a_2)u_t\left(m,\frac{m}{b_1}b_2,x,a_2\right)\right)dx$$
$$=\varphi_3(m)---------(17)$$

We give m values $m = 0,1,\ldots,n$ and (17) is interpreted as a generalized moments problem

$$\iint H_m(x,t)\Phi(x,t)dA=\mu_m$$

with

$$H_m(x,t)=\ u\left(m,\frac{m}{b_1}b_2,x,t\right)\quad m=0,1,2,\ldots,n$$

We apply the truncated expansion method and find a numerical approximation $p_{3n}(x,t)$ for $\Phi(x,t)$.

If $b_2=\infty$ we apply the truncated expansion method with

$$H_m(x,t)=\ u\left(m,\frac{m}{b_1}b_2,x,t\right)e^{-t\frac{m}{b_1}}\quad m=0,1,2,\ldots,n$$

and in principle we find an approximation for $\Phi(x,t)e^t$, in this way when orthonormalizing there is convergence when integrating.

NUMERICAL EXAMPLES

Example 1

We consider the equation

$$w_{xx}(x,t) + w_{tt}(x,t) = \Phi(x,t) \qquad E = (0,2) \times (0,3) \qquad k = 2$$

Conditions:

$$w(x,0) = \frac{4}{(3+x)^2}; \quad w(x,3) = \frac{4}{(6+x)^2}; \quad w(0,t) = \frac{4}{(3+t)^2}; \quad w(2,t) = \frac{4}{(5+t)^2}$$

$$w_t(x,0) = -\frac{8}{(3+x)^3}; \quad w_t(x,3) = -\frac{8}{(6+x)^3};$$
$$w_x(0,t) = -\frac{8}{(3+t)^3}; \quad w_x(2,t) = -\frac{8}{(5+t)^3}$$

The solution is : $w(x,t) = \frac{4}{(3+x+t)^2}$ $\qquad \Phi(x,t) = \frac{48}{(3+t+x)^4}$

Taking into account the previous theorem we calculate the accuracy as a way of comparing
$w(x,t)$ with $p_{2n}(x,t)$, and
$\Phi(x,t)$ with $p_{3n}(x,t)$

To find $p_{1n}(x,t)$ we take $n = 6$ moments $(m = 0, \ldots, 5)$
To find $p_{2n}(x,t)$ we take $n = 6$, moments $n_1 = 3, n_2 = 2$, in total $n = n_1 . n_2 = 6$ moments.

$$\sqrt{\int_0^3 \int_0^2 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.00528062$$

In the Figure 1 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x,t)$ we take $n = 9$ moments $(m = 0, \ldots, 8)$ moments.

$$\sqrt{\int_0^3 \int_0^2 \left(\Phi(x,t) - p_{3n}(x,t)\right)^2 dxdt} = 0.036196$$

In the Figure 2 we show the exact solution (light blue) and the approximate solution (magenta color).

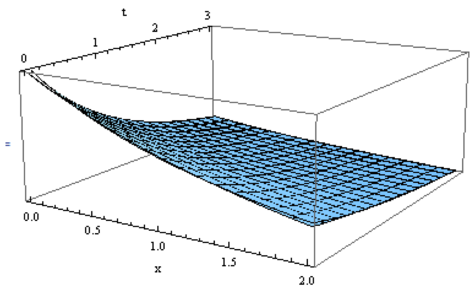

Fig. 1: $w(x,t)$ and $p_{2n}(x,t)$

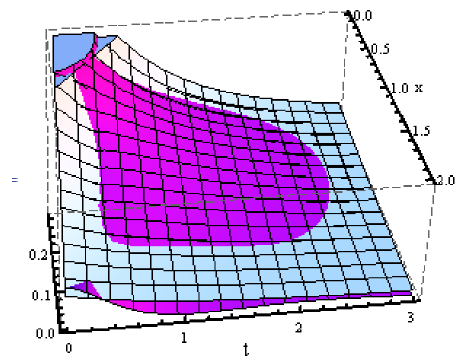

Fig. 2: $\Phi(x,t)$ and $p_{3n}(x,t)$

Example 2

We consider the equation

$$w_{tt}(x,t) + w_{xx}(x,t) = \Phi(x,t) \qquad E = (0,1) \times (0,\infty) \qquad k = 1$$

Conditions:

$$w(x,0) = e^{-1-x}; \quad w(0,t) = e^{-1-2t}; \quad w(1,t) = e^{-2-2t}$$

$$w_t(x,0) = -2e^{-1-x}; \quad w_x(0,t) = -e^{-1-2t}; \quad w_x(1,t) = -e^{-2-2t}$$

The solution is

$$w(x,t) = e^{-(x+1)-2t} \qquad \Phi(x,t) = 5e^{-1-2t-x}$$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$, and
$\Phi(x,t)$ with $p_{3n}(x,t)$

To find $p_{1n}(x,t)$ we take $n = 6$ moments $(m = 0, \dots, 5)$.
To find $p_{2n}(x,t)$ we take $n_1 = 2; n_2 = 3$, in total $n = n_1 . n_2 = 6$ moments.

$$\sqrt{\int_0^\infty \int_0^1 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.0116624$$

In the Figure 3 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x,t)$ we take $n = 9$ moments $(m = 0, \dots, 8)$ moments.

$$\sqrt{\int_0^\infty \int_0^1 \left(\Phi(x,t) - p_{3n}(x,t)\right)^2 dxdt} = 0.0402851$$

In the Figure 4 we show the exact solution (light blue) and the approximate solution (magenta color).

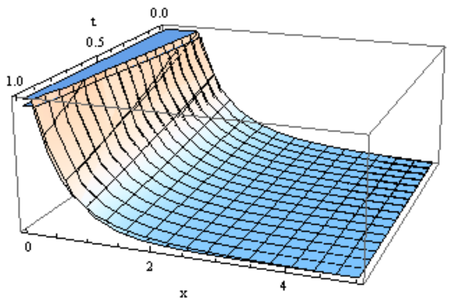
Fig. 3: $w(x,t)$ and $p_{2n}(x,t)$

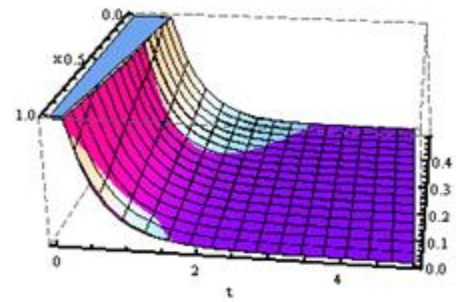
Fig. 4: $\Phi(x,t)$ and $p_{3n}(x,t)$

CONCLUSION

The problem of finding $w(x,t)$ and $\Phi(x,t)$ in the Poisson equation $w_{tt} + w_{xx} = \Phi(x,t)$ about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$, under the Cauchy conditions it is possible to solve it using the problem generalized moments techniques.
First we can deduce $w(x,t)$ in two steps.
We consider the equation $w_{xx} - kw_{tt} = -(k+1)w_{tt} + \Phi(x,t) = G(x,t)$, we turn it into an integral equation and with the techniques of inverse problem moments we find an approximate solution $p_{1n}(x,t)$ for $-\sqrt{k}w_x + kw_t$.
Then we consider the equation $-\sqrt{k}w_x(x,t) + kw_t(x,t) = p_{1n}(x,t)$ and again we take it to an integral equation to solve it and find an approximate solution $p_{2n}(x,t)$ for $w(x,t)$. Finally we consider $w_{xx}(x,t) + w_{tt}(x,t) = \Phi(x,t)$ and again we take it to an integral equation and find an approximate solution $p_{3n}(x,t)$ for $\Phi(x,t)$.

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