PAIRS OF COMPATIBLE TYPE (B) MAPPINGS AND FIXED POINT THEORY IN BANACH SPACE

[1,*]Shoyeb Ali Sayyed & [2]S.K.Jain

[1]Professor, Department of Applied Mathematics,
Lakshmi Narain College of Technology, Indore, (M.P.) India

[2]Professor, Department of Applied Mathematics,
Ujjain Engineering College Ujjain
(M.P.) India
[*]Corresponding Author email:shoyeb9291@gmail.com

Abstract

The present review article is a enhancement of fixed point theorems for two pairs of compatible type (B) mappings with two pairs of weakly compatible mappings in Banach space. The present paper divided into two parts, in first part we have generalized a common fixed point theorem for two pairs of compatible type (B) mappings and in second part we mapped a common fixed point theorem for two pairs of weakly compatible mappings in Banach space using a special type contractive condition usually named square inequality.

Keywords: *Compatible mapping of type (B), Common fixed point theorem, Weakly Compatible mapping, Square inequality, Banach space*

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1. INTRODUCTION

Compatible mapping of type (B) was initiated by Pathak and Khan [12], referring this many researchers have boosted this and further enhancement done. Jungck and Rhoades [8] introduced the notion of weakly compatible and exhibited that compatible maps are weakly compatible but converse need not be true. Using the appeal of weak compatibility as well as Compatible mapping of type (B) we have reviewed some articles and established fixed point theory. Proceeding in the same manner many authors namely Pathak [11] , Tas et,al. [18] ,Pathak and Khan [12,13] , Abbas and Rhoads [1], Ahmed and kamal [2] , Ahmed [4] Proved many theorems on various spaces using the concept of weakly compatibility and compatibility of type (B).

In this way Jain and Sayyed [6] proved Weak compatibility for four mappings and general common fixed point theorem using usual contractive type condition . Ahmed [3], Jain and Sayyed [7], Chung and Kumar [4], Lateef et. al. [9], Popa [14], Malhotra and Bansal [10],Sayyed [15] , Sayyed and Badshah [16], Ganal and Cholamjiak [5] and Sintunavart and Kumar [17] demonstrated and showed beyond doubt many theorems for it.

2. PREMILINARY

We shall use the following definitions, lemmas and theorems (without proof) for achieving our main result

Theorem 2.1[Jain and Sayyed[6]] : Let Q_1, Q_2, Q_3, and Q_4 be continuous mappings of a complete d-metric space (X, d) and satisfying,

(1) $Q_4(X) \subset Q_1(X)$ and $Q_3(X) \subset Q_2(X)$,

(2) Pairs (Q_3,Q_1) and (Q_4,Q_2) are weakly compatible and

(3) $$[d(Q_3x,Q_4y)]^2 \leq a'[d(Q_3x,Q_1x)d(Q_4y,Q_2y) + a''d(Q_1x,Q_2y) + a'''\max[d(Q_1x,Q_4y),d(Q_2y,Q_3x)] + a''''\max[d(Q_1x,Q_3x)d(Q_2y,Q_4y)]$$

where a', a'', a''' and a'''' ≥ 0 and for all x, y $\in$ X. Then Q_1,Q_2, Q_3 and Q_4 have a unique common fixed point.

Definition 2.1: [**Pathak and Khan [12]]:** Let S and T be mappings from a normed space E into itself. The mappings S and T are said to be compatible mappings of type (B) if

$$\lim_{n\to\infty} ||STx_n - TTx_n|| \leq \frac{1}{2}[\lim_{n\to\infty}||STx_n - St|| + \lim_{n\to\infty}||St - SSx_n||]$$

$$\lim_{n\to\infty}||TSx_n - SSx_n|| \leq \frac{1}{2}[\lim_{n\to\infty}||TSx_n - Tt|| + \lim_{n\to\infty}||Tt - TTx_n||]$$

whenever $\{x_n\}$ is a sequence in E such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ = t for some t $\in$ E.

Proposition 2.1[**Pathak and Khan [12]]**] Let S and T be compatible mappings of type (B) from a normed space E into itself. Suppose that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in E. \text{ then}$$

$$\lim_{n\to\infty} TTx_n = St \text{ if S is continuous at t,}$$

$$\lim_{n\to\infty} SSx_n = Tt \text{ if T is continuous at t,}$$

$$STt = TSt \text{ and } St=Tt \text{ if S and T are continuous at t.}$$

Let A and B be two mappings of a metric space (M,d) into itself. Pathak [11] defined A and B to be weakly compatible mappings with respect to B if and only if whenever,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t \text{ for some } t \in M,$$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) \leq d(At,Bt)$$

for all sequence $\{x_n\}$ in M and

$$d(At,Bt) \leq \lim_{n\to\infty} d(Bt,BAx_n)$$

for at least one sequence $\{x_n\}$in M.

The succeeding lemma is fruitful in this sequel.

Lemma 2.1. Let A,B : (M,d) → (M,d) be weakly compatible with respect to B,

(i) if At = Bt, then ABt = BAt,

(ii) suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$, for some n,

(iii) If A is continuous at t then $\lim_{n\to\infty} d(BAx_n,At) \leq d(At,Bt)$,

(iv) if A and B are continuous at t then At = Bt and ABt = BAt.

Definition 2.2 [Jungk and Rhoades [8]]: Let A and S be mappings from a metric space (*X* , *d*) into itself.Then, A and S are said to be weakly compatible if they commute at their coincident point ; that is , *Ax* = *Sx* for some *x* ϵ *X* implies *ASx* = *SAx*.

The article is divided into two sections , first section deals with compatible mappings of type (B) and second chapter deals with weakly compatible mappings by using a contractive type condition (square inequality).

3. MAIN RESULTS

SECTION - I

THEOREM 3.1 : Let G,H,I and J be mappings from Banach space X into itself and the pairs {G,I} and {H,J} are compatible of type (B), satisfying the following conditions :

$$\|Gx - Hy\|^2 \le \Phi\{ p [\|Ix - Gx\| \|Jy - Hy\| + \|Jy - Gx\| \|Ix - Hy\|] + p' [\|Ix-Hy\| \|Gx-Jy\| + \|Jy-Hy\| \|Ix-Jy\|] \} \quad \text{---} \ (A)$$

for all x, y ϵ X and p and p' are non negative with $0 \le p + p' \le 1$, and the function Φ satisfying the following conditions :

(a) $\Phi : [0,\infty) \to [0,\infty)$ is non decreasing and right continuous,

(a') for every $t > 0$, $\Phi(t) < t$ and we suppose that

$(1-k)G(X) + kI(X) \subset G(X)$, for all $k \in (0,1)$,

$(1-k^*)H(X) + k^* J(X) \subset H(X)$, for all $k^* \in (0,1)$.

For some $x_0 \in X$, the sequence $\{x_n\}$ is defined by

$$Gx_{2n+1} = (1-c_{2n}) Gx_{2n} + c_{2n} Ix_{2n} \quad \text{---} \ (A^*)$$

$$Hx_{2n+2} = (1-c_{2n+1}) Hx_{2n+1} + c_{2n+1} Jx_{2n+1} \quad \text{---} \ (A^{**})$$

with $0 < c_n \le 1$ and $\lim_{n\to\infty} c_n = h > 0$ for n = 0,1,2, ... Then $\{x_n\}$ converges to a point z in C and if G and H are continuous at z, then z is a common fixed point of G, H, I and J.

PROOF: Let z ϵ X such that $\lim_{n\to\infty} x_n = z$. Now since G is continuous at z, then we have

$Gx_n = Gz$ as $n \to\infty$. From equation (A*), we have

$$Ix_{2n} = \frac{Gx_{2n+1} - (1-c_{2n})Gx_{2n}}{c_{2n}} \to \frac{Gz-(1-h)Gz}{h} = Gz \text{ as } n \to\infty.$$

Similarly, from (A**), we can write $Jx_{2n+1} \to Hz$ as $n \to\infty$.

Now assuming $GGz \neq Hz$, then by using equation (A) with $x = Ix_{2n}$, $y = x_{2n+1}$, we obtain

$$||GIx_{2n} - Hx_{2n+1}||^2 \leq \Phi \{ p [||IIx_{2n} - GIx_{2n}|| \; ||Jx_{2n+1} - Hx_{2n+1}|| + ||Jx_{2n+1} - GIx_{2n}|| \; ||IIx_{2n} - Hx_{2n+1}||]$$

$$+ p' [||IIx_{2n}-GIx_{2n}|| \; ||GIx_{2n}-Jx_{2n+1}|| + ||Jx_{2n+1}-Hx_{2n+1}|| \; ||IIx_{2n}-Jx_{2n+1}||] \}$$

Taking limit as $n \to\infty$, we get

$$||G^2z - Hz||^2 \leq \Phi\{ p [||IGz - G^2z || \; ||Hz - Hz|| + ||Hz - G^2z || \; ||IGz - Hz||]$$

$$+ p' [||IGz-G^2z|| \; ||G^2z -Hz || + ||Hz-Hz|| \; ||IGz-Hz ||] \}$$

$$||G^2z - Hz||^2 \leq \Phi\{ p [||Hz - G^2z || \; ||IGz - Hz||] + p' [||IGz-G^2z|| \; ||G^2z -Hz ||] \}$$

Or $$||G^2z - Hz|| \leq \Phi\{ p [||IGz - Hz||] + p' [||IGz-G^2z||] \}$$

Or $$||G^2z - Hz|| \leq p \, ||G^2z - Hz||$$

Or $(1-p) ||G^2z - Hz|| \leq 0$, a contradiction, hence $GGz = Hz$.

Now suppose that $Jz \neq Gz$, then by using equation (A) and proposition 2.1, we obtain

$$||GIx_{2n} - Hz||^2 \leq \Phi\{ p [||IIx_{2n} - GIx_{2n}|| \; ||Jz - Hz || + ||Jz - GIx_{2n}|| \; ||IIx_{2n} - Hz||]$$

$$+ p' [||IIx_{2n}-GIx_{2n}|| \; ||GIx_{2n}-Jz || + ||Jz - Hz || \; ||IIx_{2n} - Jz||] \}$$

Letting $n \to\infty$, we get as $Hz = GGz$ and $||GIx_{2n} - Hx_{2n}|| \to 0$,

$$||GGz - Jz||^2 \leq \Phi\{p[||Jz - GGz||\ ||Jz - GGz||]\}$$

Or $\quad (1-p)\ ||GGz - Jz||^2 \leq 0$, a contradiction hence GGz = Jz.

Similarly we can show that HHz = Iz, therefore Gz = Hz = Iz = Jz

and $\quad IGz = I^2z = G^2z = IJz = GJz = Jz$.

So Jz = u is a common fixed point of G, H, I and J. Let v be a another common fixed point of G, H, I and J. By equation (A), we have

$$||u - v||^2 = ||Gu - Hv||^2 \leq \Phi\{ p [||Iu - Gu||\ ||Jv - Hv|| + ||Jv - Gu||\ ||Iu - Hv||] + p' [||Iu-Gu||\ ||Gu-Jv|| + ||Jv-Hv||\ ||Iu-Jv||] \}$$

Or $$||u - v||^2 \leq \Phi\{ p\ ||v - u||\ ||u - v||]$$

Or $\quad (1-p)\ ||u - v|| \leq 0$ is a contradiction hence u = v. This complete the proof.

SECTION - II

THEOREM 3.2 : Let G,H,I and J be mappings from C into itself where C be a nonempty closed convex subset of a Banach space X and satisfying the following conditions :

$$||Ix - Jy||^2 \leq \Phi\{ p [||Hx - Gx||\ ||Jy - Hy|| + ||Jy - Gx||\ ||Ix - Hy||] + p' [||Gx-Hx||\ ||Hx-Jy|| + ||Jy-Hy||\ ||Gx-Jy||] \} \quad \text{\_ \_ \_ (B)}$$

for all x, y ϵ C and p and p' are non negative with $0 \leq p + p' \leq 1$, and the function Φ satisfying the following conditions :

(a') $\Phi : [0,\infty) \rightarrow [0,\infty)$ is non decreasing and right continuous,

(a'') for every t > 0 , Φ(t) < t and we suppose that

$$(1-k)G(C) + kI(C)\ \ C\ G(C), \text{ for all } k \in (0,1) ,$$

$$(1-k^{*})H(C) + k^{*} J(C) \subset H(C), \text{ for all } k^{*} \in (0,1).$$

And the pairs (G,H), (I,H) and (J,H) are weakly compatible pairs with respect to H of X.

_ _ _ (B*)

For some $x_0 \in X$, the sequence $\{x_n\}$ is defined by

$$Gx_{2n+1} = (1-c_{2n})\, Gx_{2n} + c_{2n}\, Ix_{2n} \quad \text{--- (B**)}$$

$$Hx_{2n+2} = (1-c_{2n+1})\, Hx_{2n+1} + c_{2n+1}\, Jx_{2n+1} \quad \text{--- (B***)}$$

with $0 < c_n \leq 1$ and $\lim_{n\to\infty} c_n = h > 0$ for n = 0,1,2, ... Then $\{x_n\}$ converges to a point z in C and if G and H are continuous at z, then z is a common fixed point of G, H, I and J. Further , if G and H are continuous at z then I and J are continuous at z.

PROOF: Let $z \in C$ such that $\lim_{n\to\infty} x_n = z$. Now since G is continuous at z, then we have $Gx_n = Gz$ as $n \to\infty$. From equation (B), we have

$$Ix_{2n} = \frac{Gx_{2n+1} - (1-c_{2n})Gx_{2n}}{c_{2n}} \to \frac{Gz-(1-h)Gz}{h} = Gz \text{ as } n \to\infty.$$

Similarly, from (B*), we can write $Jx_{2n+1} \to Hz$ as $n \to\infty$.

Now assuming $Gz \neq Hz$, then by using equation (B) with $x = x_{2n}$, $y = x_{2n+1}$, we obtain

$$\|Ix_{2n} - Jx_{2n+1}\|^2 \leq \Phi \{ p [\|Gx_{2n} - Hx_{2n}\| \, \|Jx_{2n+1} - Hx_{2n+1}\| + \|Jx_{2n+1} - Gx_{2n}\| \, \|Ix_{2n} - Hx_{2n+1}\|]$$
$$+ p' [\|Gx_{2n}-Hx_{2n}\| \, \|Hx_{2n}-Jx_{2n+1}\| + \|Jx_{2n+1}-Hx_{2n+1}\| \, \|Gx_{2n}-Jx_{2n+1}\|] \}$$

Taking limit as $n \to\infty$, we get

$$\|Gz - Hz\|^2 \leq \Phi\{ p [\|Gz - Hz\| \, \|Hz - Hz\| + \|Hz - Gz\| \, \|Gz - Hz\|]$$
$$+ p' [\|Gz-Hz\| \, \|Hz -Hz\| + \|Hz-Hz\| \, \|Gz-Hz\|] \}$$

Or $\|Gz - Hz\| \leq \Phi\{ p [\|Gz - Hz\| \, \|Hz-Gz\|] \}$

Or $\|Gz - Hz\| \leq p \|Gz - Hz\|$

Or $(1-p)\ ||Gz - Hz|| \leq 0$, a contradiction, hence Gz = Hz.

Now suppose that Jz ≠ Gz, then by using equation (B) , we obtain

$$||Ix_{2n} - Jz||^2 \leq \Phi\{ p [||Gx_{2n} - Hx_{2n}|| \ ||Jz - Hz|| + ||Jz - Gx_{2n}|| \ ||Ix_{2n} - Hz||]$$
$$+ p' [||Gx_{2n} - Hx_{2n}|| \ ||Hx_{2n} - Jz|| + ||Jz - Hz|| \ ||Gx_{2n} - Jz||] \}$$

Letting $n \to \infty$, we get as Hz = Gz and $||Gx_{2n} - Ix_{2n}|| \to 0$,

$$||Gz - Jz||^2 \leq \Phi\{p'[||Jz - Gz|| \ ||Jz - Gz||]\}$$

Or $(1-p')\ ||Gz - Jz||^2 \leq 0$, a contradiction hence Gz = Jz.

Similarly we can show that Hz = Iz, therefore Gz = Hz = Iz = Jz .

From condition (B*), since the pair (G,H) is weakly compatible with respect to H and Gz = Hz, we obtain GHz = HGz by lemma 1.3. Similarly IHz = HIz and the pair (I,H) is weakly compatible with respect to H. Similarly JHz = HJz since Jz = Hz and the pair (J,H) is weakly compatible with respect to H. Hence using equation (B), we have

$$||I^2z - Jz||^2 \leq \Phi\{ p [||HIz - Gz|| \ ||Jz - Hz|| + ||Jz - GIz|| \ ||IIz - Hz||]$$
$$+ p' [||GIz-HIz|| \ ||HIz-Jz|| + ||Jz-Hz|| \ ||GIz-Jz||] \}$$

Or $||I^2z - Jz||^2 \leq \Phi\{p[||I^2z - Jz|| \ || I^2z - Jz ||]$

Or $(1-p)\ || I^2z - Jz | \leq 0$, is a contradiction. Which implies that

$$I^2z = Jz = Gz = Hz = Iz = IGz = IHz = IJz.$$

So Iz = u is a common fixed point of G, H, I and J. Let v be a another common fixed point of G, H, I and J. By equation (B), we have

$$||u - v ||^2 = ||Iu - Jv||^2 \leq \Phi\{ p [||Iu - Gu|| \ ||Jv - Hv|| + ||Jv - Gu || \ ||Iu - Hv||]$$
$$+ p' [||Iu-Gu|| \ ||Gu-Jv || + ||Jv-Hv|| \ ||Iu-Jv||] \}$$

Or $$||u - v||^2 \leq \Phi\{ p ||v - u|| ||u - v||]$$

Or $$(1-p) ||u - v|| \leq 0$$ is a contradiction hence u = v.

Now we prove that , if G and H are continuous at z, then I and J are continuous at z.

Let $\{y_n\}$ be an arbitrary sequence in C converges to z, then by equation (B), we have

$$||Iy_n - Iz||^2 = ||Iy_n - Jz||^2$$

$$\leq \Phi\{ p [||Hy_n - Gy_n|| ||Jz - Hz|| + ||Jz - Gy_n|| ||Iy_n - Hz||]$$

$$+ p' [||Gy_n-Hy_n|| ||Hy_n-Jz|| + ||Jz-Hz|| ||Gy_n-Jz||] \}$$

$$||Iy_n - Iz||^2 \leq \Phi\{ p'[||Gy_n - Gz|| ||Gy_n - Gz||]$$

Letting $n \to \infty$, we see that G is continuous , limit $n \to \infty$ $Iy_n = Iz$. thus I is continuous at z ,

similarly we can show that when H is continuous at z then J is continuous at z.

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