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CHEBYSHEV-BASED NUMERICAL MODEL AS FIRST, SECOND AND THIRD ORDER INITIAL VALUE PROBLEMS SOLVER

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Abstract

A single numerical model as integrator of initial value problems of multi-order (1st, 2nd and 3rd) ordinary differential equations is introduced. Utilizing Chebyshev polynomials as the trial function, the method is formulated firstly, by obtaining the continuous form of the proposed scheme via collocation technique and later, arrange in a block-by-block manner as numerical integrator of multi-order ODEs. The convergence properties are investigated and it's established that the proposed method is convergent. A comparison of the problems solved with the new method and existing methods shows that the new method outperformed better than existing methods in terms of accuracy.

Keywords: *Block-Method, Consistency, Convergence, Initial Value Problems, Zero-Stability*

INTRODUCTION

This paper focuses on the development of

$$\left\{\sum_{r=0}^{s}\alpha_r(t)y_{n+r} = h\left(\sum_{r=0}^{s}\beta_r(t)m_{n+r}\right) + h^2\left(\sum_{r=0}^{s}\mu_r(t)l_{n+r}\right) + h^3\left(\sum_{r=0}^{s}\delta_r(t)k_{n+r}\right)\right. \tag{1}$$

to directly integrate ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = f(x, y(x)),\ y(x_0) = y_0 \\ y''(x) = f(x, y(x), y'(x)),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \\ y'''(x) = f(x, y(x), y'(x), y''(x)),\ y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0 \end{cases} \tag{2}$$

where in (1), either of $\alpha_0(t)$ and $\beta_0(t)$ do not varnish, $\alpha_s(t)=1$, $\beta_s(t)\neq 0$ and $s=1$

Integrating (2) using block method and the resulting solutions have been extensively discussed. This block method approach which generates approximations at different grid points simultaneously without overlapping of sub-intervals has been reported to circumvent the setback commonly experienced in reducing higher-order ODEs to a first-order equations and the predictor-corrector approach. (Kayode S. & Adegboro, J. [11]).

Numerical scheme capable of handling second and third-order ODEs has been formulated by [4] using power series as the basis function while (Adeyefa E. & Kuboye J. [4]) computed a numerical integrator able to handle first, second, and third-order ODEs.

Conventionally, these methods which have been reported to be efficient are directly used as numerical integrators of ODE targeting IVPs of the same order but, to using the method to integrate initial value problems of different order has not been discussed. Thus, the focus is to foster a single numerical model for the direct numerical solution of multi-order (1st, 2nd and 3rd) ODEs.

We consider the derivation of the proposed method for direct integration of (Kayode S. & Adegboro, J. [11]) in section 2. In Section 3 the analysis of the method and its implementation are discussed and results is given in Section 4. Finally, in section 5 the conclusion of the paper is discussed.

METHOD OF SOLUTION

Interpolation and collocation technique is adopted and a new numerical scheme capable of solving multi-order (1st, 2nd and 3rd) IVPs of ODEs is formulated, employing Chebyshev polynomials as basis function.

Thus,

$$y(x)=\sum_{r=0}^{s+9}\alpha_r X_r \tag{3}$$

is an approximate solution to first, second and third-order ODEs of the form

$$y'(x)=f(x,y(x)),\quad y(x_0)=y_0 \tag{4}$$

$$y''(x)=f(x,y(x),y'(x)),\quad y(x_0)=y_0,y'(x_0)=y'_0 \tag{5}$$

$$y'''(x)=f(x,y(x),y'(x),y''(x)),\quad y(x_0)=y_0,y'(x_0)=y'_0,y''(x_0)=y''_0 \tag{6}$$

Eq. (3) is interpolated at $x = x_n$ The first and second derivatives of Eq. (3) is collocated at $x = x_{n+p}$, $p = 0, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$ and the third derivative at $x = x_{n+w}, w = 1$, we have

$$\begin{aligned}
&\sum_{r=0}^{s+9} \alpha_r X_r = y_n \\
&\sum_{r=0}^{s+9} \alpha_r X_r = m_{n+v} \\
&\sum_{r=0}^{s+9} \alpha_r X_r = l_{n+v} \\
&\sum_{r=0}^{s+9} \alpha_r X_r = k_{n+w}
\end{aligned} \tag{7}$$

where; X_r is the parameters of Chebyshev polynomials, m_{n+v} is the first derivative of (3), l_{n+v} is its second derivative, k_{n+w} is its third derivative and s is the step number (s=1).

Introducing Gaussian elimination in Eq.(7) to get the $\alpha's$ which are the unknown variables which are substituted back to Eq. (3). This yields a continuous implicit scheme of the form:

$$\alpha_0(t)y_n = h\left(\sum_{j=0}^{1} \beta_{\frac{3}{6}}(t)m_{n+\frac{3}{6}} + \beta_{\frac{4}{6}}(t)m_{n+\frac{4}{6}} + \beta_{\frac{5}{6}}(t)m_{n+\frac{5}{6}}\right) + h^2\left(\sum_{j=0}^{1} \mu_{\frac{3}{6}}(t)l_{n+\frac{3}{6}} + \mu_{\frac{4}{6}}(t)l_{n+\frac{4}{6}} + \mu_{\frac{5}{6}}(t)l_{n+\frac{5}{6}}\right) + h^3\left(\delta_1(t)k_{n+1}\right) \tag{8}$$

where, $t = \frac{2x - 2x_n - h}{h}$, $\alpha_0(t) = 1$.

$$\begin{pmatrix} \beta_0 \\ \beta_{\frac{3}{6}} \\ \beta_{\frac{4}{6}} \\ \beta_{\frac{5}{6}} \\ \lambda_0 \\ \lambda_{\frac{3}{6}} \\ \lambda_{\frac{4}{6}} \\ \lambda_{\frac{5}{6}} \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -8 & 18 & -32 & 50 & -72 & 98 & -128 & 162 \\ 0 & 2 & 0 & -6 & 0 & 10 & 0 & -14 & 0 & 18 \\ 0 & 2 & \frac{8}{3} & \frac{-10}{3} & \frac{-224}{27} & \frac{-110}{81} & \frac{920}{81} & \frac{-7826}{729} & \frac{-15232}{2187} & \frac{-13870}{29} \\ 0 & 2 & \frac{16}{3} & \frac{14}{3} & \frac{-64}{27} & \frac{-950}{81} & \frac{-1232}{81} & \frac{-5278}{729} & \frac{20224}{2187} & \frac{16898}{729} \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 & 5376 & -8640 \\ 0 & 0 & 16 & 0 & -64 & 0 & 144 & 0 & -256 & 0 \\ 0 & 0 & 16 & 32 & \frac{-64}{3} & \frac{-3040}{27} & \frac{1744}{27} & \frac{12992}{81} & \frac{187648}{729} & \frac{-10816}{243} \\ 0 & 0 & 16 & 64 & \frac{320}{3} & \frac{1600}{27} & \frac{-3280}{27} & \frac{-27776}{81} & \frac{-286976}{729} & \frac{-26240}{243} \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 21504 & 56448 & 129024 & 266112 \end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \end{pmatrix}$$

Eq. (8) is evaluated at $x = x_{n+1}(t=1)$, $x = x_{n+\frac{5}{6}}(t=\frac{2}{3})$, $x = x_{n+\frac{4}{6}}(t=\frac{1}{3})$ and $x = x_{n+\frac{3}{6}}(t=0)$. This produces the following schemes

$$\begin{pmatrix} y_{n+\frac{3}{6}} \\ y_{n+\frac{4}{6}} \\ y_{n+\frac{5}{6}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y_n + hR \begin{pmatrix} m_n \\ m_{n+\frac{3}{6}} \\ m_{n+\frac{4}{6}} \\ m_{n+\frac{5}{6}} \\ m_{n+1} \end{pmatrix} + h^2 U \begin{pmatrix} l_n \\ l_{n+\frac{3}{6}} \\ l_{n+\frac{4}{6}} \\ l_{n+\frac{5}{6}} \\ l_{n+1} \end{pmatrix} + h^3 V \begin{pmatrix} k_n \\ k_{n+\frac{3}{6}} \\ k_{n+\frac{4}{6}} \\ k_{n+\frac{5}{6}} \\ k_{n+1} \end{pmatrix} \tag{9}$$

where $R = \begin{pmatrix} \frac{5631139}{38080000} & \frac{381829}{2581875} & \frac{312815}{2115072} & \frac{176609}{1190000} \\ \frac{-44515}{7616} & \frac{-166912}{28917} & \frac{-10665625}{1850688} & \frac{-648}{119} \\ \frac{25515}{8704} & \frac{3233}{1071} & \frac{1703125}{548352} & \frac{6399}{1904} \\ \frac{15544467}{4760000} & \frac{730112}{223125} & \frac{229115}{68544} & \frac{218376}{74375} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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$$U=\begin{pmatrix} \frac{47227}{7616000} & \frac{28823}{4647375} & \frac{118075}{19035648} & \frac{1487}{238000} \\ \frac{-38377}{76160} & \frac{-653504}{1301265} & \frac{-16714375}{33312384} & \frac{-288}{595} \\ \frac{-292329}{304640} & \frac{-46451}{48195} & \frac{-4733125}{4935168} & \frac{-8289}{9520} \\ \frac{-325377}{1904000} & \frac{-206272}{1204875} & \frac{-213125}{1233792} & \frac{-1728}{14875} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{47}{228480} & \frac{268}{1301265} & \frac{6875}{33312384} & \frac{1}{3570} \end{pmatrix}$$

Analysis of the method

The analysis of basic properties of this method is investigated in this section such as consistency, error constant, order, and zero stability.

Order

Equation (9) is regarded as a scheme which belongs to the class of LMMs of the form:

$$\sum_{r=0}^{s}\alpha_r y_{n+r} = h\left(\sum_{r=0}^{s}\beta_r(t)m_{n+r}\right)+h^2\left(\sum_{r=0}^{s}\mu_r(t)l_{n+r}\right)+h^3\left(\sum_{r=0}^{s}\delta_r(t)k_{n+r}\right) \tag{10}$$

Following (Fatunla S. [8]), we define the local truncation error associated with Eq. (10) as

$$L[y(x):h]=\sum_{r=0}^{k}\left[\alpha_r y(x_n+rh)-h^2\beta_r m(x_n+rh)-h^3\gamma_r l(x_n+rh)\right] \tag{11}$$

Where L is the difference operator and $y(x)$ is continuously differentiable on the given interval.

Expanding (11) using Taylor series about the point x, the expression below is obtained

$$L[y(x);h]=C_0 y(x)+C_1 hy'(x)+C_2 h^2 y''(x)+\ldots+C_{p+2}h^{p+2}y^{p+2}(x)$$

where the C_0, $C_1, C_2 \ldots C_p \ldots C_{p+2}$ are obtained as $C_0 = \sum_{r=0}^{s} \alpha_r$, $C_1 = \sum_{r=1}^{s} r\alpha_r$, $C_2 = \frac{1}{2!}\sum_{r=1}^{s} r^2\alpha_r$,

$C_q = \frac{1}{q!}\left[\sum_{r=1}^{s} r^q\alpha_r - q(q-1)\sum_{r=1}^{s}\beta_r r^{q-2} - q(q-1)(q-2)\sum_{r=1}^{s}\gamma_r r^{q-3}\right]$.

In the spirit of (Lambert J. [14]), Eq. (11) is of order p if $C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$ and $C_{p+r} \neq 0$. The $C_{p+r} \neq 0$ is called the error constant and $C_{p+r}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the order of the block (9) is $p = 8$ and error constant

$$C_{p+2} = \left[\frac{4553}{23878316851200}, \frac{3649}{19124223249600}, \frac{398875}{2088875158142976}, \frac{337}{1678944153600}\right]^T.$$

Zero Stability of the Method

The LMMs (10) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

In analyzing the zero-stability of the method, equation (10) is presented in vector notation form of column vector $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_j)^T$, $y_m = (y_{n+1} \ldots y_{n+j})^T$, $M(y_m) = (m_{n+1} \ldots m_{n+j})^T$, $L(y_m) = (l_{n+1} \ldots l_{n+j})^T$, $K(y_m) = (k_{n+1} \ldots k_{n+j})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, Eq. (9) forms the block formula

$$A^0 y_m = hRV(y_m) + A^1 y_n + hrv_n + h^2UL(y_m) + h^2Ul_n + h^3MW(y_m) + h^3uK_n \tag{12}$$

where h is a fixed mesh size within a block.

In accordance with (12),

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, w = \begin{pmatrix} \frac{5631139}{38080000} \\ \frac{-44515}{7616} \\ \frac{25515}{8704} \\ \frac{15544467}{4760000} \\ 0 \end{pmatrix}, q = \begin{pmatrix} \frac{47227}{7616000} \\ \frac{-38377}{76160} \\ \frac{-292329}{304640} \\ \frac{-325377}{1904000} \\ 0 \end{pmatrix}, z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{47}{228480} \end{pmatrix}$$

$$W = \begin{pmatrix} \frac{381829}{2581875} & \frac{312815}{2115072} & \frac{176609}{1190000} \\ \frac{-166912}{28917} & \frac{-10665625}{1850688} & \frac{-648}{119} \\ \frac{3233}{1071} & \frac{1703125}{548352} & \frac{6399}{1904} \\ \frac{730112}{223125} & \frac{229115}{68544} & \frac{218376}{74375} \\ 0 & 0 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} \frac{28823}{4647375} & \frac{118075}{19035648} & \frac{1487}{238000} \\ \frac{-653504}{1301265} & \frac{-16714375}{33312384} & \frac{-288}{595} \\ \frac{-46451}{48195} & \frac{-4733125}{4935168} & \frac{-8289}{9520} \\ \frac{-206272}{1204875} & \frac{-213125}{1233792} & \frac{-1728}{14875} \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{268}{1301265} & \frac{6875}{33312384} & \frac{1}{3570} \end{pmatrix}$$

$$\rho(R) = \det(RA^0 - A^1) \tag{13}$$

Equation (13) is the first characteristic polynomial of the block hybrid method

Where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

substituting A^0 and A^1 in equation (13) and solving for R, the values of R are obtained as 0 ,0,0 and 1.

In the Spirit of (Lambert J. [14]), equation (9) is zero-stable, since from [10], $\rho(R) = 0$, satisfy $|R_j| \le 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

Consistency and convergence of the method

If linear multistep method (10) has order $p \geq 1$, it is said to be consistent. Hence the method is consistent being of order p=8

The sufficient and necessary condition for a LMMs to be convergent is to be zero stable and consistent, see (Dahlquist G. [7]). Hence, the method is convergent.

NUMERICAL RESULTS AND DISCUSSION:

Differential problems which include first, second and third-order ordinary differential equations are implemented to test the effectiveness of the new scheme formulated in this section.

Problem 1: $y' = 0.5(1-y), \;\; y(0) = 0.5, h = 0.1$

Exact Solution: $y(x) = 1 - 0.5e^{-0.5x}$

Table 1: Comparing the error of the new block method with existing methods for solving Problem 1

x-values	Error in new method,	Error in [4]	Error in [5]	Error in [21]
0.1	0.000000E+000	1.000000E-10	1.218026E-13	5.574430E-12
0.2	1.000000E-20	1.000000E-10	1.399991E-13	3.946177E-12
0.3	0.000000E+000	1.000000E-10	1.184941E-12	8.183232E-12
0.4	0.000000E+000	2.000000E-10	1.538991E-12	3.436118E-15
0.5	1.0000000E-20	3.000000E-10	1.110001E-12	1.929743E-10
0.6	0.000000E+000	3.000000E-10	5.270229E-12	1.879040E-10
0.7	0.000000E+000	2.000000E-10	2.10898E-12	1.776835E-10
0.8	0.000000E+000	3.000000E-10	1.297895E-11	1.724676E-10
0.9	0.000000E+000	3.000000E-10	3.08229E-11	1.847545E-10
1.0	0.000000E+000	2.000000E-10	4.121925E-11	3.005770E-10

Problem 2: $y''' = 3\sin x, \;\; y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$

Exact Solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 2*:* Error comparison of the new block method with existing methods for solving Problem 2

x-values	Error in the new method	Error in [4]	Error in [12]	Error in [17]
0.1	7.000000E-020	2.000000E-010	6.370460E-13	1.65922E-10
0.2	1.400000E-019	4.000000E-010	4.052980E-12	4.76275E-10
0.3	1.400000E-019	2.000000E-010	1.009326E-11	6.23182E-10
0.4	2.300000E-019	2.000000E-010	1.890366E-11	19.9134E-10
0.5	3.300000E-019	9.000000E-010	3.033807E-11	3.28882E-10
0.6	3.300000E-019	1.100000E-009	4.455258E-11	1.27096E-09
0.7	4.100000E-019	1.500000E-009	5.987466E-11	4.84653E-09
0.8	4.600000E-019	1.300000E-009	7.711903E-11	1.09585E-08
0.9	4.100000E-019	1.500000E-009	9.618412E-11	2.0188E-08
1.0	5.000000E-019	2.000000E-009	1.171654E-10	3.53956E-08

Problem3 : $y'' = y', \quad y(0) = 0, y'(0) = -1, \; h = 0.1$

Exact Solution: $y(x) = 1 - e^{x}$

Table 3*:* Comparing the error of the new block method with existing methods for solving Problem 3

x-values	Error in the new method	Error in [3]	Error in [12]	Error in [12]
0.1	4.200000E-020	2.095826E-010	2.508826E-13	2.858824E-15

0.2	-4.200000E-020	2.092718E-009	6.493175E-11	1.439682E-12
0.3	-1.200000E-020	7.842546E-009	1.683146E-09	5.591383E-11
0.4	-4.300000E-020	2.009500E-008	1.700635E-08	4.796602E-09
0.5	-2.000000E-020	4.199771E-008	1.025454E-07	1.003781E-08
0.6	2.500000E-020	7.728842E-008	2.558711E-06	1.590163E-08
0.7	-4.700000E-020	1.303844E-007	5.273300E-06	2.870014E-08
0.8	3.600000E-020	2.064839E-007	8.275935E-06	4.284730E-08
0.9	-2.700000E-020	3.116817E-007	1.161667E-05	5.857869E-08
1.0	1.000000E-020	4.531001E-007	1.542187E-05	8.449297E-08

Problem 4: $y''' = e^x \quad y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1$

Exact Solution: $y(x) = 2 + 2x^2 + e^x$

Table 4: Comparing the error of the new block method with existing methods for solving Problem 4

x-values	Error in the new method	Error in [3]	Error in [12]	Error in [17]
0.1	0.000000E+000	8.881784E-015	3.369305E-12	9.24352E-10
0.2	0.000000E+000	3.552714E-014	2.160050E-11	8.3983E-10
0.3	0.000000E+000	8.304468E-014	5.333245E-11	4.23997E-10
0.4	-1.000000E-019	1.527667E-013	9.988632E-11	3.58729E-10
0.5	-2.000000E-019	2.460254E-013	1.598988E-10	2.99872E-10
0.6	-2.000000E-019	3.668177E-013	2.511404E-10	3.90509E-10
0.7	-3.000000E-019	5.178080E-013	3.961489E-10	1.47048E-09
0.8	-3.000000E-019	7.025491E-013	5.926823E-10	2.49247E-09
0.9	-4.000000E-019	9.254819E-013	8.429168E-10	0.15695E-09
1.0	-4.000000E-019	1.187495E-012	1.144603E-09	3.54096E-09

Problem 5 : $y''' = y', \quad y(0) = 0, y'(0) = -1, h = 0.1$

Exact Solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Table 5*:* Comparing the error of the new block method with existing methods for solving Problem 5

x-values	Error in the new method,	Error in [19]	Error in [12]
0.1	0.00000E+000	1.194048000E-013	2.508826E-13
0.2	0.00000E+000	4.086842000E-013	6.493175E-11
0.3	1.00000E-019	1.016689500E-012	1.683146E-09
0.4	0.00000E+000	2.139483600E-012	1.700635E-08
0.5	0.00000E+000	4.083580200E-012	1.025454E-07
0.6	0.00000E+000	7.350069300E-012	2.558711E-06
0.7	-1.00000E-019	1.279204250E-012	5.273300E-06

CONCLUSION

In this paper, a single numerical model developed has been used in solving multi-order ordinary differential equations directly. The method is consistent because is of order 8.The advantage of the new method over existing methods is that it is efficient in handling different orders of differential equations namely first, second and third-order ordinary differential equations. The efficiency of the new method is proved by applying it to first, second and third-order ordinary differential equations, from the results generated, the new method outperformed the existing methods in terms of error as shown in Tables I – V

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