OPTIMAL CONTROL ANALYSIS OF COMPUTER VIRUS EPIDEMIC MODEL

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Abstract

In this paper, a model for the transmission dynamics of computer virus is formulated and two control strategies: a strong antivirus and treatment are deployed to minimize the total number of infected computers and the cost associated with the control strategies. Optimal control theory is applied to a system of ordinary differential equations of a computer virus epidemic. The Pontryagin's maximum principle is employed to find the necessary and sufficient conditions for the existence of the optimal controls. Runge-Kutta forward-backward sweep numerical approximation method is used to solve the optimal control system. The results show that for the computer virus spread to be under control in the area where it is epidemic, 50% of the strong antivirus and 50% of treatment should be continuously implemented.

Keywords: *SPEIQRS Model, transmission, Computer, Virus, Pontryagin's maximum principle, optimal control, antivirus, treatment, Hamiltonian, numerical simulation*.

INTRODUCTION

A computer virus is a program which can harm our device and files and infect them for no further use. When a virus program is executed, it replicates itself by modifying other computer programs and instead enters its own coding. This code infects a file or program and if it spreads massively, it may ultimately result in crashing of the device (Chinebu et. al., 2021).

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function (performance index) is optimized (Ross, 2015). The optimal control (OC) can also be defined as the process of determining the control and state trajectories for a dynamic system over a period of time in order to minimize a performance index. It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function. It can be derived using Pontryagin's maximum principle (a necessary condition also known as Pontryagin's minimum principle or simply Pontryagin's Principle) (Ross, 2009) or by solving the Hamiltonian equation for sufficient condition.

The application of optimal control theory has become another interesting area in the field of mathematical modeling (Lenhart, 2007) in that it provides insightful understanding of many biomedical problems and its being used extensively in the controlling of infectious diseases (Lenhart, 2007).

Again the work of (Neilan and Lenhart, 2010) serves as an introduction to the theory of optimal control applied to systems of ordinary differential equations with emphasis on disease models. They outline the steps in formulating an optimal control problem and derive necessary conditions. Several simple examples are provided with a detailed methodology in characterizing the optimal control through use of Pontryagin Maximum Principle.

The Hamiltonian is a function used to solve a problem of optimal control for a dynamical system. It can be seen as an instantaneous increment of the Lagrangian expression of the problem that is to be optimized over a certain time horizon (Ferguson and Lim, 1998). The Hamiltonian of optimal control theory was developed by Lev Pontryagin as part of his maximum principle (Dixit, 1990). Pontryagin proved that a necessary condition for solving the optimal control problem is that the control should be chosen so as to optimize the Hamiltonian (Kirk, 1970).

In the Optimal Control Problem, we introduced two control strategies: $u_1 = u_1(t)$ and $u_2 = u_2(t)$ to represent a strong antivirus on susceptible computers and treatment on infected computers respectively. Control u_1 is meant to protect the susceptible computer when interacting with the removable media or infectious computers. Control u_2 is meant to treat the infectious computers (Infected and quarantined computers).

The goal is to find an optimal control policy function $u^*(t)$ and with it, an optimal trajectory of the state variable $y^*(t)$ which by Pontryagin's maximum principle are the arguments that maximize the Hamiltonian,

$$H\left(y^*(t), u^*(t), \lambda(t), t\right) \geq H\left(y(t), u(t), \lambda(t), t\right) \text{ for all } u(x) \in U \tag{1}$$

(a) The necessary conditions for optimality (Kamien and Schwartz, 1991)

The first-order necessary conditions for a maximum optimality are:

i. $$\frac{\partial H\left(y(t), u(t), \lambda(t), t\right)}{\partial u} = 0 \quad \text{(Optimal control equation)} \tag{2}$$

ii. $$\frac{d\lambda}{dt} = -\frac{\partial H\left(y(t), u(t), \lambda(t), t\right)}{\partial y} \quad \text{(Optimal co-state equation)} \tag{3}$$

iii. $$\lambda(T) = \left.\frac{\partial \varphi}{\partial y}\right|_{t_f} = 0 \quad \text{(Transversality condition)} \tag{4}$$

(b) The sufficient condition for the optimality

i. maximum optimality is the concavity of the Hamiltonian

$$\frac{\partial^2}{\partial u^2} H\left(y^*(t), u^*(t), \lambda(t), t\right) \leq 0 \text{ at } u^*(t) \text{ or} \tag{5}$$

ii. minimum optimality is the convexity of the Hamiltonian

$$\frac{\partial^2}{\partial u^2} H\left(y^*(t), u^*(t), \lambda(t), t\right) > 0 \text{ at } u^*(t) \text{ and} \tag{6}$$

where $u^*(t)$ is the optimal control and $y^*(t)$ is the resulting optimal trajectory for the state variable (Seierstad and Sydsalter, 1987; Lenhart and Workman 2007). Alternatively, the necessary conditions are sufficient if the functions $L\left(y(t), u(t), t\right)$ and $f\left(y(t), u(t), t\right)$ are both concave in $y(t)$ and $u(t)$ (Mangasarian, 1966).

SOME LITERATURE REVIEW OF OPTIMAL CONTROL MODEL OF THE COMPUTER VIRUS

From 1987 to 1988, F. Cohen and W. Murray discovered certain similarities between computer viruses and biological infectious diseases (Cohen, 1987; Murray, 1988). Consequently, they suggested applying the principles of infectious disease dynamics and qualitative and quantitative analysis methods to study the patterns of computer virus transmission. Unfortunately, they did not propose specific models for the spread of computer viruses at that time. It was not until 1991 that J. O. Kephart and S. R. White adopted the recommendations of F. Cohen and W. Murray (Kephart and White, 1991). Based on the similarity between computer viruses and biological viruses, they introduced the SIS (susceptible–infected–susceptible) computer virus propagation model for the first time, pioneering the application of biological virus propagation models to the field of computer viruses. Since then, extensive research has been conducted on computer viruses.

Kazeem et al, (2017) in their study formulated a deterministic computer virus model, incorporating removal devices. They studied the basic properties of the model, calculated the reproduction number and the steady state which was found to be stable. Time optimal control was included and Pontryagin’s Maximum Principle was used to characterize all the necessary conditions for controlling the spread of computer virus. They showed that the most effective strategy for controlling computer virus is through the combination of the three controls, that is, intensive public education on the use of removable devices, public campaign on computer virus free and treatment of infected computers.

On the optimal control of a malware propagation model, (Guillen et. al., 2014), stated that the important way to control malware epidemic process is to take into account the security measures that are associated to the systems of ordinary differential equations that govern the dynamic of such systems. Here they considered two types of control measures, that is the analysis of the basic reproduction number and the study of control measure functions. They used the theory of optimal control that is associated to the systems of ordinary equations in order to find a new function to control malware epidemic through time.

MATHEMATICAL FORMULATION OF OPTIMAL CONTROL PROBLEM OF THE MODEL

The optimal control model for computer virus is formulated in order to derive two optimal control measures with minimal implementation cost to eradicate the virus with a strong antivirus after a limited period of time. The control efforts $u_i(t); i = 1,2$ are employed in human and animal reservoirs populations and $\left(1 - u_i(t)\right)$ are the failure rate for the control efforts for $i = 1,2$. Let $u_1(t)$ be the effort of using a strong antivirus as a control measure

ASSUMPTIONS

The following model assumptions are made.
We introduced

i. a strong antivirus to the computer system at a rate u_1 such that $u_1S(t)$ computers per time are removed from the susceptible computer class.
ii. treatment to both the infected and quarantined computers at a rate u_2 such that $u_2I(t)$ and $u_2Q(t)$ computers per time are removed from the infected and quarantined class respectively.

THE SCHEMATIC FLOW DIAGRAM OF THE MODEL WITH CONTROL VARIABLES

We demonstrate the dynamical transfer of the population with the flow diagram in Figure1 below.

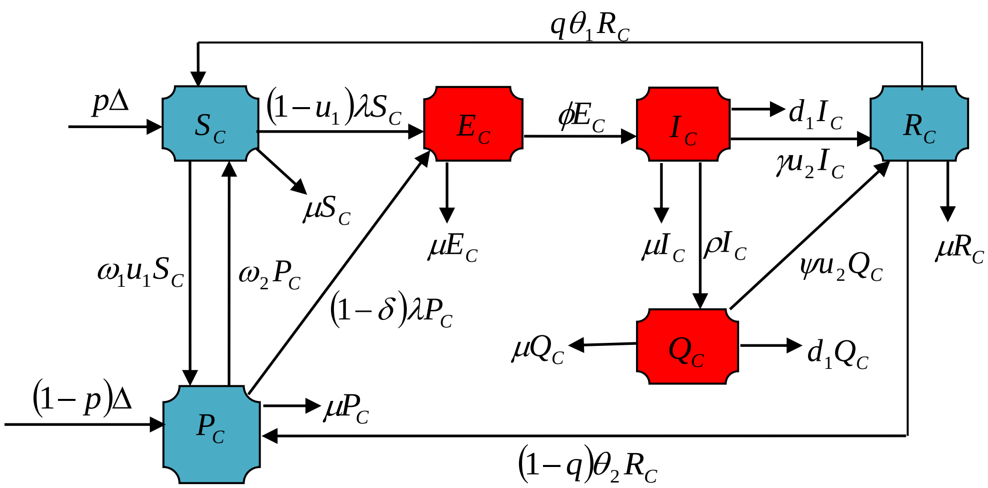

Figure 1: A schematic representation of flow diagram of computers virus spread among the states variables with control variables.

OPTIMAL CONTROL VARIABLES OF THE MODEL

u_1 = The effort of using a strong antivirus to the computer system as a control measure by installing a strong antivirus in the system.
u_2 = The effort of using treatment as a control measure by treating the infected computers.

Table 1: DESCRIPTION OF THE VARIABLES OF THE MODEL

Variables	Description	Values	Reference
$S_C(0)$	Number of susceptible computers at time (t).	65	(Shahrear *et al.*, 2018)
$P_C(0)$	Number of protected computers at time (t).	20	(Shahrear *et al.*, 2018)
$E_C(0)$	Number of exposed computers at time (t).	10	(Shahrear *et al.*, 2018)
$I_C(0)$	Number of infected computers at time (t).	6	Assumed

$Q_C(0)$	Number of quarantined computers at time (t).	2	Assumed
$R_C(0)$	Number of Recovered individuals at time (t).	5	(Shahrear *et al.*, 2018)

Table 2: DESCRIPTION OF THE PARAMETERS OF THE MODEL

Parameters	Description	Values	Reference
Δ	The recruitment rate.	0.30	(Udegbe *et al.*, 2021)
w_1	The rate at which the susceptible computers moved to the protected computer class.	0.65	(Shahrear *et al.*, 2018)
w_2	The rate at which the protected computers moved to the susceptible computer class.	0.35	Assumed
μ	The death rate of computers due to other factors that are not connected to computer virus.	0.10	(Udegbe *et al.*, 2021)
d_1	The death rate of computers due to computer virus.	0.65	(Udegbe *et al.*, 2021)
ϕ	The rate at which the exposed computers moved to the infected computer class.	0.20	(Shahrear *et al.*, 2018)
p	The proportion of the recruited computers that moved to the susceptible class due none installation of antivirus at a rate Δ.	0.90	Assumed
$(1-p)$	The proportion of the recruited computers that moved to the protected class due to the installation of a strong antivirus at a rate Δ.	0.10	Assumed
θ_1	The rate at which the recovered computers moved to the susceptible computer class.	0.01	(Shahrear *et al.*, 2018)
θ_2	The rate at which the recovered computers moved to the protected computer class.	0.05	Assumed
q	The proportion of the recovered computers that moved to the susceptible class at a rate θ_1.	0.75	Assumed
$(1-q)$	The proportion of the recovered computers that moved to the protected class at a rate θ_2.	0.25	Assumed
ψ	The rate at which the quarantined computers moved to the recovered computer class.	0.03	Assumed
ρ	The rate at which the infected	0.02	(Udegbe *et al.*, 2021)

	Computers moved to the quarantined class.		
β_1	The transmission rate of computer virus to the infected computers due to the transfer of files from one computer to another using flash or other means.	0.09	(Shahrear *et al.*, 2018)
β_2	The transmission rate of computer virus to the quarantined computers due to the transfer of files from one computer to another using flash or other means.	0.05	Assumed
$1-\delta$	The proportion of the protected computers that moved to the exposed class due to the fact that no antivirus offers computers 100% protection .	0.35	(Shahrear *et al.*, 2018)
γ	The rate at which the infected computers moved to the recovered computer class.	0.45	(Shahrear *et al.*, 2018)

RESULTANT EQUATIONS OF THE MODEL WITH CONTROL VARIABLES

The computer virus is modeled as a system of ordinary differential equations with the two control variables embedded within the dynamical system as stated below:

$$\frac{dS_C}{dt} = p\Delta - (1-u_1)(\beta_1 I_C + \beta_2 Q_C)S_C - w_1 u_1 S_C - \mu S_C + w_2 P_C + q\theta_1 R_C \tag{1}$$

$$\frac{dP_C}{dt} = (1-p)\Delta + w_1 u_1 S_C - (w_2 + \mu)P_C - (1-\delta)(\beta_1 I_C + \beta_2 Q_C)P_C + (1-q)\theta_2 R_C \tag{2}$$

$$\frac{dE_C}{dt} = (1-u_1)(\beta_1 I_C + \beta_2 Q_C)S_C + (1-\delta)(\beta_1 I_C + \beta_2 Q_C)P_C - (\phi + \mu)E_C \tag{3}$$

$$\frac{dI_C}{dt} = \phi E_C - \gamma u_2 I_C - (\rho + d_1 + \mu)I_C \tag{4}$$

$$\frac{dQ_C}{dt} = \rho I_C - \psi u_2 Q_C - (d_1 + \mu)Q_C \tag{5}$$

$$\frac{dR_C}{dt} = \gamma u_2 I_C + \psi u_2 Q_C - q\theta_1 R_C - (1-q)\theta_2 R_C + \mu R_C \tag{6}$$

$$N_C = S_C + P_C + E_C + I_C + Q_C + R_C \tag{7}$$

The objective functional is defined as follows

$$J(u_1, u_2, u_3, u_4, u_5) = \int_0^T \left(A_1 I_C + A_2 Q_C + \frac{1}{2}\left(B_1 u_1^2 + B_2 u_2^2\right)\right) dt \tag{8}$$

subject to

$$\left.\begin{aligned}
\frac{dS_C}{dt} &= p\Delta - (1-u_1)(\beta_1 I_C + \beta_2 Q_C)S_C - w_1 u_1 S_C - \mu S_C + w_2 P_C + q\theta_1 R_C \\
\frac{dP_C}{dt} &= (1-p)\Delta + w_1 u_1 S_C - (w_2 + \mu)P_C - (1-\delta)(\beta_1 I_C + \beta_2 Q_C)P_C + (1-q)\theta_2 R_C \\
\frac{dE_C}{dt} &= (1-u_1)(\beta_1 I_C + \beta_2 Q_C)S_C + (1-\delta)(\beta_1 I_C + \beta_2 Q_C)P_C - (\phi + \mu)E_C \\
\frac{dI_C}{dt} &= \phi E_C - \gamma u_2 I_C - (\rho + d_1 + \mu)I_C \\
\frac{dQ_C}{dt} &= \rho I_C - \psi u_2 Q_C - (d_1 + \mu)Q_C \\
\frac{dR}{dt} &= \gamma u_2 I_C + \psi u_2 Q_C - q\theta_1 R_C - (1-q)\theta_2 R_C + \mu R_C
\end{aligned}\right\} \tag{9}$$

with initial conditions

$S_C(0) = S_0 > 0$, $P_C(0) = P_0 \geq 0$, $E_C(0) = E_0 \geq 0$, $I_C(0) = I_0 \geq 0$, $Q_C(0) = Q_0 \geq 0$, $R_C(0) = R_0 \geq 0$, $u_1(0) = u_{10}$, $u_2(0) = u_{20}$ and with control set defined as;

$$U = \left\{\begin{matrix} (u_1(t), u_2(t)) : 0 \leq u_1(t) \leq u_{1\max} \leq 1,\ 0 \leq u_2(t) \leq u_{2\max} \leq 1 \\ \\ \text{are Lebesgue measureable and piecewise continous on } [0, \mathrm{T}] \end{matrix}\right\} \tag{10}$$

where $(1-u_1)(\beta_1 I_C + \beta_2 Q_C)$ is the reduction in transmission due to infected computers and quarantined; A_1 and A_2 are the weight constants or balance factor (i.e. to keep balance) in the size of infected computers and treatment population respectively (Zaman *et al.*, 2008) , whereas B_1 and B_2 are the weights of control variables $u_i(t)$ relative to its cost implications for a strong virus and treatment efforts respectively which regulate the optimal control, the square term of the controls indicates the nonlinearity in the cost of controls, and the half-term minimizes the effect of applying the controls (Asamoah *et al.*, 2017). Also, the positive constants B_1 and B_2 which are connected with $u_i(t)$ are weight values such that $0 < B_1, B_2 < N$. The relative weights given to the positive constants associated with the control terms indicate greater or lower importance placed on minimizing the cost of a control measure (Anita *et al.*, 2011). We also assume that the cost of a strong antivirus and treatment is non linear and quadratic as seen in the cost function (8). $B_1 u_1^2$ represents the cost of a strong antivirus and $B_2 u_2^2$ represents the cost of treating the infected computers.

The main goal is to characterize an optimal control $(u_1^*, u_2^*) \in U$ which minimizes the cost of a strong antivirus and treatment as well as minimizing the number of infectives at terminal time (T) such that the number of susceptible computers decreases. Therefore, to obtain the optimal solutions, we defined the Lagrangian L for the control problem as follows:

$$L(N,u) = A_1 I_C + A_2 Q_C + \frac{1}{2}\left(B_1 u_1^2 + B_2 u_2^2\right), \tag{11}$$

where N is the total population of computers. Thus, the Hamiltonian function H for the control problem is defined as:

$$H(N,u;t)=A_1I_C+A_2Q_C+\frac{1}{2}B_1u_1^2+\frac{1}{2}B_2u_2^2+\lambda_1\frac{dS_C}{dt}+\lambda_2\frac{dP_C}{dt}+\lambda_3\frac{dE_C}{dt}+\lambda_4\frac{dI_C}{dt}+\lambda_5\frac{dQ_C}{dt}+\lambda_6\frac{dR_C}{dt} \quad (12)$$

$$\begin{aligned}H(N,u;t)=&A_1I_C+A_2Q_C+\frac{1}{2}\left(B_1u_1^2+B_2u_2^2\right)\\&+\lambda_1\left[p\Delta-(1-u_1)(\beta_1I_C+\beta_2Q_C)S_C-w_1u_1S_C-\mu S_C+w_2P_C+q\theta_1R_C\right]\\&+\lambda_2\left[(1-p)\Delta+w_1u_1S_C-(w_2+\mu)P_C-(1-\delta)(\beta_1I_C+\beta_2Q_C)P_C+(1-q)\theta_2R_C\right]\\&+\lambda_3\left[(1-u_1)(\beta_1I_C+\beta_2Q_C)S_C+(1-\delta)(\beta_1I_C+\beta_2Q_C)P_C-(\phi+\mu)E_C\right]\\&+\lambda_4\left[\phi E_C-\gamma u_2I_C-(\rho+d_1+\mu)I_C\right]\\&+\lambda_5\left[\rho I_C-\psi u_2Q_C-(d_1+\mu)Q_C\right]\\&+\lambda_6\left[\gamma u_2I_C+\psi u_2Q_C-q\theta_1R_C-(1-q)\theta_2R_C+\mu R_C\right]\end{aligned} \quad (13)$$

MODEL ANALYSIS

OPTIMAL CONTROL SOLUTION

We apply the Pontryagin's Maximum Principle (PMP) (Pontryagin *et al.*, 1962) because it is a constrained control problem. The principle identifies the adjoint functions of the optimal system and represents an optimal control in terms of the state and adjoint functions. The main goal of this principle (PMP) is to minimize the objective function. Thus depending on the constraints in the objective function, we want to minimize the Hamiltonian with respect to the controls u_1 and u_2. We define the adjoint functions as λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 associated with state equations defined for S_C, P_C, E_C, I_C, Q_C, R_C.

Theorem 11: (Pontryagin et al., 1962)**:** Given that $U^*(t)=\left(u_1^*,u_2^*\right)$ is an optimal control and that $X^*(t)=\left(S^*(t),P^*(t),E^*(t),I^*(t),Q^*(t),R^*(t)\right)$ is an optimal state solution for the dynamical optimal control problem (13) that minimize $J(u_1,u_2)$ over U, there exist adjoint variables $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\lambda_4(t)$, $\lambda_5(t)$ and $\lambda_6(t)$ that satisfies the dynamical model below.

$$\frac{d\lambda_1}{dt}=(\lambda_1-\lambda_3)(\beta_1I_C+\beta_2Q_C)(1-u_1)+(\lambda_1-\lambda_2)w_1u_1+\lambda_1\mu$$

$$\frac{d\lambda_2}{dt}=(\lambda_2-\lambda_3)(1-\delta)(\beta_1I_C+\beta_2Q_C)+(\lambda_2-\lambda_1)w_2+\lambda_2\mu$$

$$\frac{d\lambda_3}{dt}=(\lambda_3-\lambda_4)\phi+\lambda_3\mu$$

$$\frac{d\lambda_4}{dt}=-A_1+(\lambda_1-\lambda_3)(1-u_1)\beta_1S_C+(\lambda_2-\lambda_3)(1-\delta)\beta_1P_C+(\lambda_4-\lambda_6)\gamma u_2+(\lambda_4-\lambda_5)\rho+\lambda_4(d_1+\mu)$$

$$\frac{d\lambda_5}{dt}=-A_2+(\lambda_1-\lambda_3)(1-u_1)\beta_2S_C+(\lambda_2-\lambda_3)(1-\delta)\beta_2P_C+(\lambda_5-\lambda_6)\psi u_2+\lambda_5(d_1+\mu)$$

$$\frac{d\lambda_6}{dt}=(\lambda_6-\lambda_1)q\theta_1+(\lambda_6-\lambda_2)(1-q)\theta_2+\lambda_6\mu$$

with transversality conditions: $\lambda_1(T)=0$ and $\lambda_2(T)=0$ and control functions (u_1^*,u_2^*) which satisfies the optimality condition given by:

$$u_1^* = \min\left\{\max\left\{0,\left[\frac{(\lambda_3-\lambda_1)(\beta_1 I_C+\beta_2 Q_C)+(\lambda_1-\lambda_2)w_1}{B_1}\right]S_C\right\},1\right\} \tag{14}$$

$$u_2^* = \min\left\{\max\left\{0,\left[\frac{(\lambda_4-\lambda_6)\gamma I_C+(\lambda_5-\lambda_6)\psi Q_C}{B_2}\right]\right\},1\right\} \tag{15}$$

Proof:

In order to obtain the co-state dynamical model and the transversality conditions, we need to apply the Pontryagin's maximum principle (PMP) (Pontryagin *et al.*, 1962) to partially differentiate the Hamiltonian function H with respect to the corresponding state variables S_C, P_C, E_C, I_C, Q_C and R_C,.The proofs go as follows:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial S_C} = \lambda_1[(\beta_1 I_C+\beta_2 Q_C)(1-u_1)-w_1u_1+\mu]+\lambda_2[-w_1u_1]+\lambda_3[-(\beta_1 I_C+\beta_2 Q_C)(1-u_1)]\\
&= (\lambda_1-\lambda_3)(\beta_1 I_C+\beta_2 Q_C)(1-u_1)+(\lambda_1-\lambda_2)w_1u_1+\lambda_1\mu\\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial P_C} = \lambda_1(-w_2)+\lambda_2[(w_2+\mu)+(1-\delta)(\beta_1 I_C+\beta_2 Q_C)]+\lambda_3[-(1-\delta)(\beta_1 I_C+\beta_2 Q_C)]\\
&= (\lambda_2-\lambda_3)(1-\delta)(\beta_1 I_C+\beta_2 Q_C)+(\lambda_2-\lambda_1)w_2+\lambda_2\mu\\
\frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial E_C} = \lambda_3(\phi+\mu)+\lambda_4[-\phi]\\
&= (\lambda_3-\lambda_4)\phi+\lambda_3\mu\\
\frac{d\lambda_4}{dt} &= -\frac{\partial H}{\partial I_C} = -A_1+\lambda_1[\beta_1(1-u_1)S_C]+\lambda_2[(1-\delta)\beta_1 P_C]+\lambda_3[-(1-u_1)\beta_1 S_C-(1-\delta)\beta_1 P_C]\\
&\quad +\lambda_4[\gamma u_2+(\rho+d_1+\mu)]+\lambda_5[-\rho]+\lambda_6[-\gamma u_2]\\
&= -A_1+(\lambda_1-\lambda_3)(1-u_1)\beta_1 S_C+(\lambda_2-\lambda_3)(1-\delta)\beta_1 P_C+(\lambda_4-\lambda_6)\gamma u_2+(\lambda_4-\lambda_5)\rho+\lambda_4(d_1+\mu)\\
\frac{d\lambda_5}{dt} &= -\frac{\partial H}{\partial Q_C} = -A_2+\lambda_1[\beta_2(1-u_1)S_C]+\lambda_2[(1-\delta)\beta_2 P_C]+\lambda_3[-(1-u_1)\beta_2 S_C-(1-\delta)\beta_2 P_C]\\
&\quad +\lambda_5[\psi u_2+(d_1+\mu)]+\lambda_6[-\psi u_2]\\
&= -A_2+(\lambda_1-\lambda_3)(1-u_1)\beta_2 S_C+(\lambda_2-\lambda_3)(1-\delta)\beta_2 P_C+(\lambda_5-\lambda_6)\psi u_2+\lambda_5(d_1+\mu)\\
\frac{d\lambda_6}{dt} &= -\frac{\partial H}{\partial R_C} = \lambda_1[-q\theta_1]+\lambda_2[-(1-q)\theta_2]+\lambda_6[q\theta_1+(1-q)\theta_2+\mu]\\
&= (\lambda_6-\lambda_1)q\theta_1+(\lambda_6-\lambda_2)(1-q)\theta_2+\lambda_6\mu
\end{aligned}$$

Again, applying the necessary conditions from Pontryagin's maximum principle, we know that u^* must be a critical point of the Hamiltonian, that is, $\frac{\partial H}{\partial u_i}=0$. By differentiating the Hamiltonian function H w.r.t $u_i(t)$, we obtain $u_1^*(t)$ and $u_2^*(t)$respectively.

$$\frac{\partial H}{\partial u_1} = B_1 u_1 + \lambda_1\left[\left(\beta_1 I_C + \beta_2 Q_C\right)S_C - w_1 S_C\right] + \lambda_2\left[w_1 S_C\right] + \lambda_3\left[-\left(\beta_1 I_C + \beta_2 Q_C\right)S_C\right] = 0 \quad (16)$$

$$B_1 u_1 = \left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right)S_C + \left(\lambda_1 - \lambda_2\right)w_1 S_C$$

$$u_1^* = \frac{\left[\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1\right]S_C}{B_1} \quad (17)$$

$$\frac{\partial H}{\partial u_2} = B_2 u_2 + \lambda_4\left[-\gamma I_C\right] + \lambda_5\left[-\psi Q_C\right] + \lambda_6\left[\gamma I_C + \psi Q_C\right] = 0 \quad (18)$$

$$B_2 u_2 = \left(\lambda_4 - \lambda_6\right)\gamma I_C + \left(\lambda_5 - \lambda_6\right)\psi Q_C$$

$$u_2^* = \frac{\left(\lambda_4 - \lambda_6\right)\gamma I_C + \left(\lambda_5 - \lambda_6\right)\psi Q_C}{B_2} \quad (19)$$

with transversality conditions: $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$. The characterization of the optimal controls $u_1^*(t)$ and $u_2^*(t)$ are based on the conditions: $\frac{\partial H}{\partial u_1} = \frac{\partial H}{\partial u_2} = 0$ subject to the constraints: $0 \le u_1(t) \le u_{1\max} \le 1$, $0 \le u_2(t) \le u_{2\max} \le 1$, (Modnak, 2013).

Using the bounds of the $u_i(t)$, its optimal control can be written in compact form as:

$$u_1^* = \min\left\{\max\left\{0, \left[\frac{\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1}{B_1}\right]S_C\right\}, 1\right\} \quad (20)$$

$$u_2^* = \min\left\{\max\left\{0, \left[\frac{\left(\lambda_4 - \lambda_6\right)\gamma I_C + \left(\lambda_5 - \lambda_6\right)\psi Q_C}{B_2}\right]\right\}, 1\right\} \quad (21)$$

Therefore, from equations (20) and (21), we obtain the following optimality system

$$\begin{aligned}\frac{dS_C}{dt} = p\Delta &- \left\{1 - \min\left\{\max\left\{0, \left[\frac{\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1}{B_1}\right]S_C\right\}, 1\right\}\right\}\left(\beta_1 I_C + \beta_2 Q_C\right)S_C \\ &- w_1\left\{\min\left\{\max\left\{0, \left[\frac{\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1}{B_1}\right]S_C\right\}, 1\right\}\right\}S_C - \mu S_C + w_2 P_C \\ &+ q\theta_1 R_C\end{aligned} \quad (22)$$

$$\begin{aligned}\frac{dP_C}{dt} = (1-p)\Delta &+ w_1\left\{\min\left\{\max\left\{0, \left[\frac{\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1}{B_1}\right]S_C\right\}, 1\right\}\right\}S_C - \left(w_2 + \mu\right)P_C \\ &- (1-\delta)\left(\beta_1 I_C + \beta_2 Q_C\right)P_C + (1-q)\theta_2 R_C\end{aligned} \quad (23)$$

$$\begin{aligned}\frac{dE_C}{dt} = &\left\{1 - \min\left\{\max\left\{0, \left[\frac{\left(\lambda_3 - \lambda_1\right)\left(\beta_1 I_C + \beta_2 Q_C\right) + \left(\lambda_1 - \lambda_2\right)w_1}{B_1}\right]S_C\right\}, 1\right\}\right\}\left(\beta_1 I_C + \beta_2 Q_C\right)S_C - (\phi + \mu)E_C \\ &+ (1-\delta)\left(\beta_1 I_C + \beta_2 Q_C\right)P_C\end{aligned} \quad (24)$$

$$\frac{dI_C}{dt} = \phi E_C - \gamma\left\{\min\left\{\max\left\{0, \left[\frac{\left(\lambda_4 - \lambda_6\right)\gamma I_C + \left(\lambda_5 - \lambda_6\right)\psi Q_C}{B_2}\right]\right\}, 1\right\}\right\}I_C - \left(\rho + d_1 + \mu\right)I_C \quad (25)$$

$$\frac{dQ_C}{dt} = \rho I_C - \psi\left\{\min\left\{\max\left\{0, \left[\frac{(\lambda_4 - \lambda_6)\gamma I_C + (\lambda_5 - \lambda_6)\psi Q_C}{B_2}\right]\right\}, 1\right\}\right\} Q_C - (d_1 + \mu) Q_C \tag{26}$$

$$\frac{dR_C}{dt} = (\gamma I_C + \psi Q_C) u_2 - q\theta_1 R_C - (1-q)\theta_2 R_C + \mu R_C$$

$$\frac{dR_C}{dt} = (\gamma I_C + \psi Q_C)\left\{\min\left\{\max\left\{0, \left[\frac{(\lambda_4 - \lambda_6)\gamma I_C + (\lambda_5 - \lambda_6)\psi Q_C}{B_2}\right]\right\}, 1\right\}\right\} - q\theta_1 R_C - (1-q)\theta_2 R_C + \mu R_C \tag{27}$$

$$\frac{d\lambda_1}{dt} = (\lambda_1 - \lambda_3)\left\{1 - \min\left\{\max\left\{0, \left[\frac{(\lambda_3 - \lambda_1)(\beta_1 I_C + \beta_2 Q_C) + (\lambda_1 - \lambda_2) w_1}{B_1}\right] S_C\right\}, 1\right\}\right\}(\beta_1 I_C + \beta_2 Q_C) + (\lambda_1 - \lambda_2) w_1 \left\{\min\left\{\max\left\{0, \left[\frac{(\lambda_3 - \lambda_1)(\beta_1 I_C + \beta_2 Q_C) + (\lambda_1 - \lambda_2) w_1}{B_1}\right] S_C\right\}, 1\right\}\right\} + \lambda_1 \mu \tag{28}$$

$$\frac{d\lambda_2}{dt} = (\lambda_2 - \lambda_3)(1-\delta)(\beta_1 I_C + \beta_2 Q_C) + (\lambda_2 - \lambda_1) w_2 + \lambda_2 \mu \tag{29}$$

$$\frac{d\lambda_3}{dt} = (\lambda_3 - \lambda_4)\phi + \lambda_3 \mu \tag{30}$$

$$\frac{d\lambda_4}{dt} = -A_1 + (\lambda_1 - \lambda_3)\left\{1 - \min\left\{\max\left\{0, \left[\frac{(\lambda_3 - \lambda_1)(\beta_1 I_C + \beta_2 Q_C) + (\lambda_1 - \lambda_2) w_1}{B_1}\right] S_C\right\}, 1\right\}\right\}\beta_1 S_C + (\lambda_2 - \lambda_3)(1-\delta)\beta_1 P_C + (\lambda_4 - \lambda_6)\gamma\left\{\min\left\{\max\left\{0, \left[\frac{(\lambda_4 - \lambda_6)\gamma I_C + (\lambda_5 - \lambda_6)\psi Q_C}{B_2}\right]\right\}, 1\right\}\right\} + (\lambda_4 - \lambda_5)\rho + \lambda_4 (d_1 + \mu) \tag{31}$$

$$\frac{d\lambda_5}{dt} = -A_2 + (\lambda_1 - \lambda_3)\left\{1 - \min\left\{\max\left\{0, \left[\frac{(\lambda_3 - \lambda_1)(\beta_1 I_C + \beta_2 Q_C) + (\lambda_1 - \lambda_2) w_1}{B_1}\right] S_C\right\}, 1\right\}\right\}\beta_2 S_C + (\lambda_2 - \lambda_3)(1-\delta)\beta_2 P_C + (\lambda_5 - \lambda_6)\psi\left\{\min\left\{\max\left\{0, \left[\frac{(\lambda_4 - \lambda_6)\gamma I_C + (\lambda_5 - \lambda_6)\psi Q_C}{B_2}\right]\right\}, 1\right\}\right\} + \lambda_5 (d_1 + \mu) \tag{32}$$

$$\frac{d\lambda_6}{dt} = (\lambda_6 - \lambda_1) q\theta_1 + (\lambda_6 - \lambda_2)(1-q)\theta_2 + \lambda_6 \mu \tag{33}$$

To determine the convexity of the system (that is to check whether the critical point is actually minimum), that is $\frac{\partial^2 H}{\partial u_i^2} > 0$ (Mangasarian, 1966), we differentiated (16) and (18) w.r.t. $u_i; \mathrm{i} = 1,2$ and it resulted to $\frac{\partial^2 H}{\partial u_1^2} = B_1 > 0$, $\frac{\partial^2 H}{\partial u_2^2} = B_2 > 0$, where B_1 and B_2 are nonnegative constant.

NUMERICAL SIMULATIONS OF THE OPTIMAL CONTROL MEASURE IN THE COMPUTER VIRUS MODEL WITH RESULTS AND INTERPRETATION

In this section, we use Runge-Kutta fourth order method to approximate the optimal solutions. To solve the optimality system with initial conditions for the states and final time conditions for the adjoints, we use the Runge-Kutta fourth order procedure. A Runge-Kutta method is a multiple-step method, where the solution at time t_{k+1} is obtained from a defined set of previous values $t_{n+k}, \ldots, t_k$ and n is the number of steps.

The numerical simulation conducted in order to investigate the effects of the control strategies on the transmission dynamics of Computer Virus. The simulations are performed using MALAB and we set time in days. The estimated initial values of the state variables of the model are $S_0 = 65, P_0 = 20, E_0 = 10, I_0 = 6, Q_0 = 2$ and $R_0 = 5$ and for the adjoint system we have terminal conditions $\{\lambda_i(T)\,\forall\, i = 1,2,3,4,5,6\} = 0$, where we set $T = 50$ days. The cost coefficients corresponding to state variables are estimated to be $A_1 = 0.01$ and $A_2 = 0.02$. The quadratic cost coefficient corresponding to control measures is estimated to be $B_1 = 100$ and $B_2 = 50$. The numerical simulation of the optimality system was performed using the parameter values given in Table 2.

The graphs were plotted to show the effects of the control strategies: strong antivirus and treatment in eradicating Computer Virus infection. But we first plot the graphs when no control strategy is implemented and then compare their results with when the control strategies are implemented as shown in Figures 2(a)–2(c) in order to determine the best option to control the spread for maximum benefit.

THE GRAPH OF WITH AND WITHOUT OPTIMAL CONTROL STRATEGIES AND THEIR RESPECTIVE INTERPRETATIONS

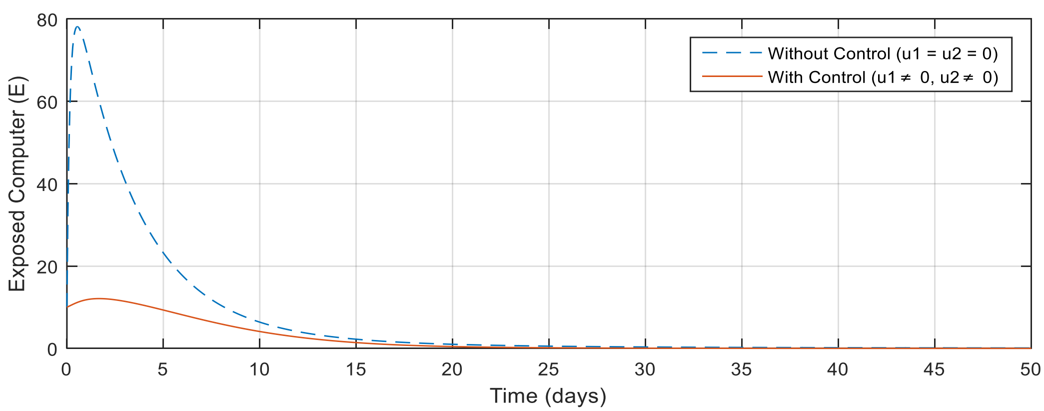

Figure 2(a) Comparison: Numerical solutions for the exposed computer virus with and without optimal control functions.

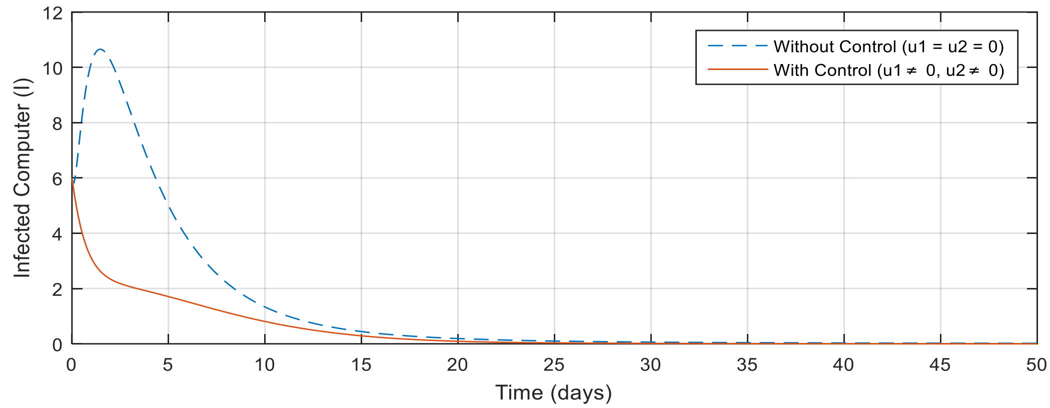

Figure 2(b) Comparison: Numerical solutions for the infected computer virus with and without optimal control functions.

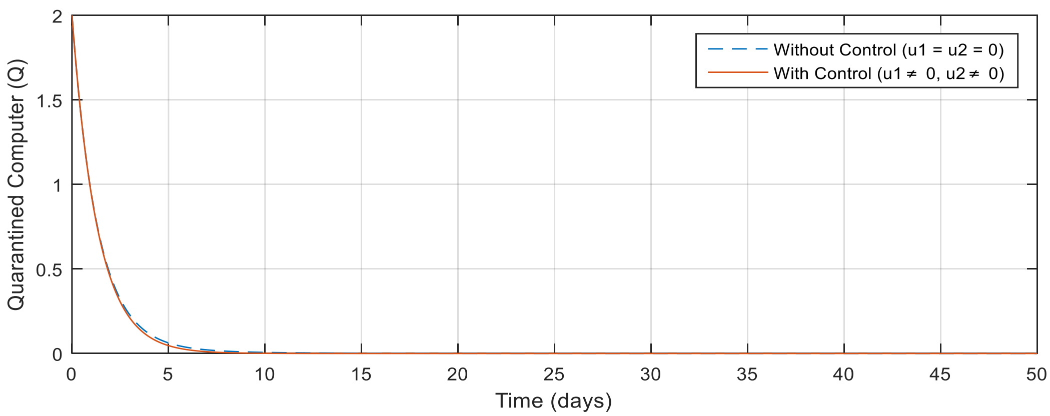

Figure 2(c) Comparison: Numerical solutions for the quarantined computer virus with and without optimal control functions.

It can be observed from figures 2(a), 2(b) and 2(c) that with the application of optimal control strategies: strong antivirus and treatment, there is a significant reduction in the number of exposed, infected and quarantined computer virus at a given time. In the same vein, the application of the optimal control strategies helps to minimize computer virus spread of the exposed computer after $t = 17$ days, infected computer after $t = 17$ days and bacteria population after $t = 6$ days. With the application of two controls to contain computer virus with respect to the result above, eradication of computer virus is possible at the shortest period of time.

THE CONTROL PROFILE OF THE COMPUTER VIRUS OF THE MODEL

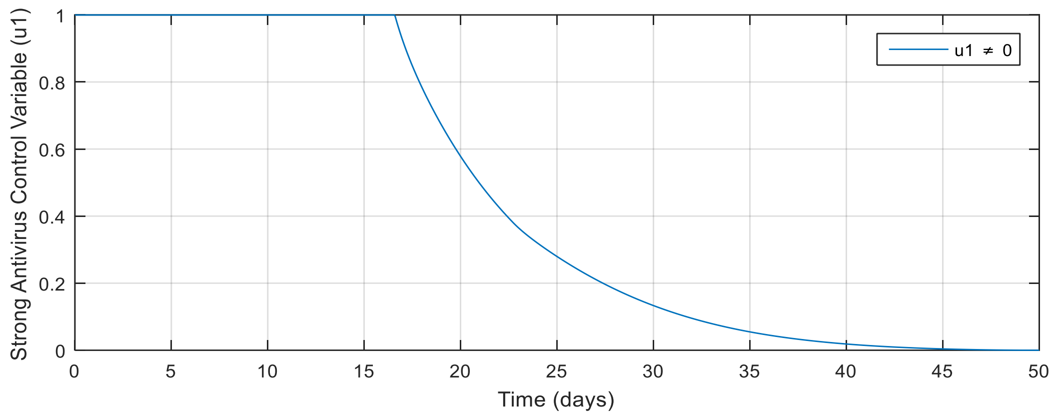

Figure 3(a): (The graph of u_1)

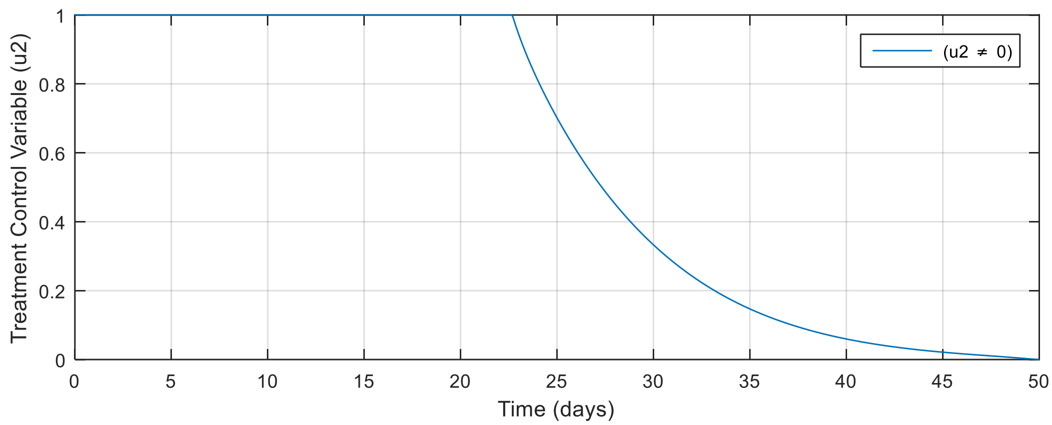

Figure 3(b): (The graph of u_2)

INTERPRETATION OF THE CONTROL PROFILE

It can be observed that Figure (3a) and Figure (3b) are individual simulation of the time-dependent optimal controls (u_1, u_2). The control u_1 is at maximum level (i.e. $u_{1\max} = 0.9999$) at $t = 0$ and remains constant for $t = 16.55$ days before it gradually drops to its minimum level (i.e. $u_{1\min} = 0$) at the final time $t = 50$ days. The control u_2 is at maximum level (i.e. $u_{2\max} = 0.9999$) at $t = 0$ and remains constant for $t = 22.65$ days before it sharply drops to its minimum level (i.e. $u_{2\min} = 0$) at the final time $t = 50$ days.

THE ANALYSIS AND INTERPRETATION OF THE GRAPH OF THE CONTROL VALUES

The controls u_1 and u_2 were used to optimize the objective function and the following observations were made:

The control values are:

$u_1 = 0.9999$

$u_2 = 0.9999$

Proportion of $u_1 = \frac{u_1}{u_1 + u_2}$

$$u_1 = \frac{0.9999}{1.9998} \times 100 = 50\%$$

Proportion of $u_2 = \frac{u_2}{u_1 + u_2}$

$$u_2 = \frac{0.9999}{1.9998} \times 100 = 50\%$$

CONCLUSION

Considering the result of study, we conclude that for the optimal control of computer virus spread to be under control in the community, 50% of a strong antivirus and 50% of treatment should be continually implemented.

REFERENCES

1. Anita, S., Capasso, V. and Arnautu, V. (2011). An Introduction to Optimal Control problems in Life Sciences and Economics: From Mathematical Models to Numerical Simulation with MATLAB®, Springer, Berlin, Germany.
2. Asamoah, J. K. K., Oduro, F. T., Bonyah, E. and Seidu, B. (2017). "Modelling of rabies transmission dynamics using optimal control analysis," *Journal of Applied Mathematics*, vol. 2017, Article ID 2451237, 23 pages. View at: Publisher Site | Google Scholar.
3. Chen H, Huang Y (2014). Study of computer virus propagation models in networks based on the stability and control. Bitechnology an Indian Journal. 10(13):7416-7421.
4. Chinebu TI, Udegbe IV, Eberendu (2021) AC. for the control of malware in computer network. Journal of Advances in Mathematics and Computer Science. 36(3):72-96.
5. Chinebu, T. I., Udegbe, I. V., & Ezennorom, E. O. (2021). Analysis of optimal control strategies for preventing computer virus infection and reduce program files damage with other symptoms. *Journal of Scientific Research and Reports*, 8-29. https://doi.org/10.9734 /jsrr/2021/v27i730408.
6. Cohen, F. (1987). Computer viruses: Theory and experiments. *Comput. Secur.6*, 22–35.
7. Dixit, A. K. (1990). Optimization in Economic Theory. New York: Oxford University Press. pp. 145–161. ISBN 978-0-19-877210-1.
8. Ferguson, B. S. and Lim, G. C. (1998). Introduction to Dynamic Economic Problems. Manchester: Manchester University Press. pp. 166–167. ISBN 0-7190-4996-2.
9. Fleming WH, Rishel RW (1975). Deterministic and stochastic optimal control. New York: Springer Verlag.
10. Guillen JDH, Del Ray AM, Vara RC (2014). On the optimal control of a malware propagation model. Mathematics. 8:1518.
11. Kazeem O, Samson O, Ebenezer B (2017) Optimal control analysis of computer virus transmission . International Journal of Network Security. 20(5):971-982.
12. Kephart, J.O. and White, S.R. (1991).Directed-graph Epidemiological Models of Computer Viruses. In Proceedings of the IEEE Computer Society Symposium on Research in Security and Privacy, Oakland, CA, USA, pp. 343–359.

13. Kirk and Donald E. (1970). *Optimal Control Theory : An Introduction*. Englewood Cliffs: Prentice Hall. p. 232. ISBN 0-13-638098-0.
14. Lenhart, S. and Workman, J.T. (2007). Optimal Control Applied to Biological Models. Chapman & Hall/CRC.
15. Mangasarian, O. L. (1966). "Sufficient Conditions for the Optimal Control of Nonlinear Systems". SIAM Journal on Control. **4** (1): 139–152. doi:10.1137/0304013.
16. Murray, W. (1988). The application of epidemiology to computer viruses. *Comput. Secur.7*, 139–150.
17. Pontryagin, L.S. and Boltyanskii, V.G., et al., (1962) The Mathematical Theory of Optimal Processes. Interscience Publishers John Wiley & Sons, Inc., New York-London, Translated from the Russian by K.N., Trirogoff.
18. Ross, I. M. (2009). A Primer on Pontryagin's Principle in Optimal Control. Collegiate Publishers. ISBN 978-0-9843571-0-9.
19. Ross, I. (2015). A primer on Pontryagin's principle in optimal control. San Francisco: Collegiate Publishers. ISBN 978-0-9843571-0-9. OCLC 625106088.
20. Shahrear P, Chakraborty AK, Islam MA, Habiba U. Analysis of computer virus propagation based on compartmental model. Applied and Computational Mathematics. 2018;7(1-2):12-21. DOI: 10.1164/j.acm.s.2018070102.12.
21. Rachael Neilan, Suzanne Lenhart (2010). An introduction to optimal control with an application in disease modeling. DIMACS Series in Dicrete Mathematics and Theoretical Computer Science.
22. Zaman G., Han Kang Y. and Jung I. H. (2008). “Stability analysis and optimal vaccination of an SIR epidemic model,” *BioSystems*, vol. 93, no. 3, pp. 240–249. View at: Publisher Site | Google Scholar.