APPLICATION OF GRAPHICAL REVISED FUZZY METRIC SPACES TO FRACTIONAL DIFFERENTIAL EQUATIONS WITH FIXED POINT THEOREMS

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Abstract:

We presented the notion of graphical revised fuzzy metric spaces in this paper, which is a relation-based expansion of revised fuzzy metric spaces. We also explored topological structure and convergence requirements, as well as proving a Banach fixed-point result in graphical revised fuzzy metric space. In the framework of graphical revised fuzzy metric spaces, we establish a solution to an integral equation and nonlinear fractional differential equations using the presented results. We've included several examples to assist you visualize the results.

Keywords: *revised fuzzy metric space, fixed point,* ***graphical*** *revised fuzzy metric spaces*

Introduction

Graph theory is becoming more popular as a result of its applications. It is widely used in mathematics and applied sciences, where it has a significant impact. A graph W = (V(W), E(W)) is made up of an ordered pair, with V(W) representing the set of all vertices of a graph W and E(W) representing the connection between the vertices of the graph W, also known as a set of edges. In addition, jVj represents the cardinality of vertex set V (also known as graph W's order) and jEj represents the cardinality of graph W's edge set E. The main graph theory is divided into two categories: undirected graphs, in which there is no direction or constraint for moving one vertex to the other if all vertices are linked, and directed graphs, in which each edge has a specific direction. A directed graph's edges are ordered pairs, whereas an undirected graph's edges are unordered pairs. The "Father of Graph Theory," Leonhard Eular

(1736), solved the famous unresolved issue known as the Konigsberg Bridge Problem. Data flow diagrams, decision-making abilities, the ability to explain links between objects, and the ability to readily adapt and change existing systems are all examples of applications due to the graphical aspect of this theory. Chemical sciences [3,25], data science [13], and architecture [27] all use graph theory in some way.

On the other hand, Jachymski [9] applied graphical structure to fixed-point theory and used it to verify various fixed-point results. With an application to integral equations, Shukla et al. [26] employed graphical structure in metric spaces to generalise numerous fixed-point findings in graphical metric space. With application, Chen et al. [4] presented a graphical convex metric space and fixed-point findings for set valued G-contraction mappings.

In 1965, fuzzy set theory initiated by L. A. Zadeh [28] introduced a method to deal with uncertainty and vagueness in our daily life. Kramosil and Michalek [12] used the notions of fuzzy sets with metric spaces established the notion of fuzzy metric spaces (FMSs) via continuous t-norms (CTNs), in which the notion of probabilistic metric spaces is generalized. George and Veeramani [7, 8] modified the concept of FMSs and obtained a Hausdorff topology for this kind of FMSs. Moreover, the theory of fixed points on FMS was developed in many directions, generalizing the fuzzy metric space or using different type of operators [6,14-16]. Alexander Sostak [1] developed the concept of "Revised Fuzzy Metrics(RFM)". Muraliraj and Thangathamizh [17-20] extended it to the notion of revised fuzzy contraction (RFC) and prove a theorem on fixed point in revised fuzzy metric space (RFMS).

We shall broaden the concept of RFMSs in this paper. The following are the manuscript's key goals:

(a) to develop a new concept of GRFMSs;

(b) to compare ordered RFMS with GRFMS;

(c) to make numerous contraction mappings and fixed point findings more universal;

(d) to determine the presence of an integral equation solution using fixed point theory;

(e) Fixed point theory is used to determine the presence of a solution to fractional differential equations.

Some Basics from Graph Theory

In this part, we go over some basic graph theory terminology that will help readers comprehend this work. The authors cite the articles of Jachymski [9] and Shukla et al. [26] for further information.

Consider a non-empty set X, and assume that X^2 is the cartesian product; the diagonal of X^2 is denoted by L. Assume W is a direct graph with non-parallel edges, such that the set of its vertices $V(W)$ corresponds with X and all loops are included in the set of its edges $E(W)$; we call X endowed with graph $W = (V(W), E(W))$ if $L \subseteq E(W)$. W^{-1} is the graph's W conversion, which is defined as:

$$V(W^{-1}) = V(W) \text{ and } E(W^{-1}) = \{(v, \theta) \in X^2 : (\theta, v) \in E(W)\}.$$

W denotes the undirected graph analysed from W, which contains all of W^{-1} edges. We write;

$$V(\widehat{W}) = V(W) \text{ and } E(\widehat{W}) = V(W) \cup E(W^{-1}).$$

Suppose v, θ are the vertices of a graph W; then the path having length $l \in N$ from $v\ to\ \theta$ in W is a sequence $\{v_i\}_{i=0}^{l}$ of $l+1$ vertices such that $v_0 = v, v_l = \theta$ and $(v_{i-1}, v_i) \in E(W)$ for $i = 1,2,3, \dots, l$. If there exists a path between any two vertices, then we say that W is connected. If there exists a path from $v\ to\ \theta$ and $\theta\ to\ v$ between two vertices in a direct graph, then we say that vertices $v\ and\ \theta$ are connected. If every one of its edges are dealt as being undirected and there is a path between every two vertices, then we say that W is weakly connected. Alternatively, if $\widehat{W}$ is connected, then W is weakly connected.

We define a relation P on X by $(vP\theta)_W$ if and only if there is a direct path from $v\ to\ \theta$ in W. The authors say $w \in (vP\theta)_W$ if w is included in a direct path from $v\ to\ \theta$ in W. For all $l \in N$, the authors define;

$$[v]_W^l = \{\theta \in X : there\ is\ a\ direct\ path\ from\ v\ to\ \theta\ of\ length\ l\}$$

A sequence $\{v\}_n$ in X is known as W −termwise connected if it satisfies $(v_n P v_{n+1})_W$ for all $n \in N$. A connected component of an undirected graph is a sub-graph that has any two vertices connected to each other by pathways and is not connected to anymore vertices in the super-graph.

In this paper, we allot a revised fuzzy weight $U_W(v, \theta, t)$ to a degree of nearness between v and θ at t, where t is interpreted as a time. We assume that the graphs under study are directed with non-empty sets of vertices and edges throughout this paper.

Definition 2.1.

Suppose X be a non-empty set endowed with a graph $(W, \oplus)$ is a continuous t-conorm, and U_W is a RFM on $X^2 \times (0, +\infty)$; the mapping U_W is known as a graphical revised fuzzy metric $briefly, GRFM$), if

a. $U_W(v, \theta, t) < 1$;

b. $U_W(v, \theta, t) = 0$ if and only if $v = \theta$;

c. $U_W(v, \theta, t) = U_W(\theta, v, t)$;

d. $if\ (vP\theta)_W$ such that $w \in (vP\theta)_W$ implies $U_W(v, \theta, t + s) \leq U_W(v, w, t) \oplus U_W(w, \theta, s)$;

e. $U_W(v, \theta, -): (0, +\infty) \to [0,1]$ is continuous, for all $v, \theta \in X$ and $t, s > 0$. Then $(X, U_W, \oplus)$ is known as a GRFMS.

Example 2.2.

Every RFMS is a GRFMS with a graph W, where $V(W) = X$ and $E(W) = X^2$ at a time t.

Definition 2.3.

Let $(X, \preccurlyeq)$ be a partially ordered set, $\oplus$ is a continuous t-conorm, and $U_{\preccurlyeq}$ is a RFM on $X^2 \times (0, +\infty)$; the mapping $U_{\preccurlyeq}$ is known as ordered RFM, if

1. $U_{\preccurlyeq}(v, \theta, t) < 1$;
2. $U_{\preccurlyeq}(v, \theta, t) = 0$ if and only if $v = \theta$;
3. $U_{\preccurlyeq}(v, \theta, t) = U_{\preccurlyeq}(\theta, v, t)$;
4. $U_{\preccurlyeq}(v, \theta, t + s) \leq U_{\preccurlyeq}(v, w, t) \oplus U_{\preccurlyeq}(w, \theta, s)$;
5. $U_W(v, \theta, -): (0, +\infty) \to [0,1]$ is continuous, for all $v, \theta \in X$ and $t, s > 0$. Then $(X, U_{\preccurlyeq}, \oplus)$ is known as a GRFMS.

Example 2.4.

Let $(X, U_{\preccurlyeq}, \oplus)$ be an ordered RFMS and W_I be a graph defined by $V(W_I) = X$ and $E(W_I) = \{(v, \theta) \in X^2 : V \preccurlyeq \theta\}$ at a time t. Then $\left(X, U_{W_I}, \oplus\right)$ is a GRFMS, where $V_{W_I}(v, \theta, t) = U_{\preccurlyeq}(v, \theta, t)$ for all $(v, \theta) \in X$ and $t > 0$. Therefore, each ordered RFMS is a graphical

RFMS with the given graph W_I, Where W_I is an induced graph by the partial order $\preccurlyeq$.

Example 2.5.

Let $X = N$ define CTCN by $a \oplus b = max\{a, b\}$ and partial order by $\preccurlyeq := \{(v, \theta) \in N \times N :\ \theta\ is\ divisible\ by\ v\}$. Define $U_{\preccurlyeq}$ by:

$$U_{\preccurlyeq}(v,\theta,t)=\begin{cases}0 & if\ v=\theta\\ \dfrac{v\theta}{t+v\theta} & if\ otherwise\end{cases}$$

Then $U_{\preccurlyeq}$ is an ordered RFM and $(X,U_{\preccurlyeq},\oplus)$ is an ordered RFMS.

Then the GRFM $V_{W_I}(v,\theta,t)=U_{\preccurlyeq}(v,\theta,t)$ is not a RFM. Indeed, for $t=s=0, v=5, \theta=1$ and $w=3$, then we obtain:

$$U_W(v,w,t+s)=\frac{vw}{t+s+vw}=\frac{15}{17}$$

On the other Side:

$$U_W(v,\theta,t)\oplus U_W(\theta,w,s)=max\left\{\frac{v\theta}{t+v\theta},\frac{w\theta}{s+w\theta}\right\}=max\left\{\frac{5}{6},\frac{3}{4}\right\}=\frac{5}{6}$$

Then we have , $U_{\preccurlyeq}(v,w,t+s)=\frac{15}{17}>\frac{5}{6}=U_{\preccurlyeq}(v,\theta,t)\oplus U_{\preccurlyeq}(\theta,w,s)$

Example 2.6.

Suppose an undirected graph $W(V,E)$ with $V(W)=X$. Let $W_j, j=1,2,\dots,n$ be the components of W that are connected such that $V(W)=\bigcup_{j=1}^{n}V(W_j)$, $E(W)=\Lambda\cup\left\{\bigcup_{j=1}^{n}V(V_j)\right\}$ at a time t. let $l_{u\theta}$ be the length of the shortest from u to $u\ to\ \theta$. Now define $U_W: X^2\times(0,+\infty)\to[0,1]$ with $a\oplus b=a+b-ab$ by

$$U_W(v,\theta,t)=\begin{cases}0 & if\ v=\theta\ or\ v\in U_j,\theta\in U_i, i,j\in\{1,2,\dots,n\}, i\neq j;\\ \dfrac{v\theta}{t+v\theta} & if\ (u,\theta)\in U_j, j\in\{1,2,\dots,n\}\end{cases}$$

Then U_W is a GRFM on X and $(X,U_W,\oplus)$ is an GRFMS.

Example 2.7.

Suppose $X=[0,1]$ and the graph $W(V,E)$ described $V(W)=X$ and $E(W)=\Lambda\cup\{(v,\theta)\in X^2: u,\theta\in(0,1], u\leq\theta\}$ at a time t. let $a\oplus b=max\{a,b\}$ and described a mapping $U_W: X^2\times(0,+\infty)\to[0,1]$ by

$$U_W(v,\theta,t)=\begin{cases}0 & if\ v=\theta;\\ \dfrac{v\theta}{t+v\theta} & if\ (v,\theta]\in(0,1], u\neq\theta\\ \dfrac{(v+\theta)}{t+(v+\theta)} & if\ otherwise\end{cases}$$

Then, U_W is GRFM on X and $(X,U_W,\oplus)$ is a GRFMS. Indeed, U_W is not an RFM on X. Let $v=\frac{1}{2}$, $\theta=\frac{1}{2}$, $u=1$ and $t=s=1$, therefore deducing that, As seen in the above examples, every RFMS is a GRFM is a GRFMS, but the converse is not true.

Definition 2.8.

Let $(X, U_W, \oplus)$ be a GRFMS. An open ball $B_W(v, r, t)$ with center $v \in X$, radius $r > 0$ is defined by

$B_W(v, r, t) = \{\theta \in X : (vP\theta)_W, U_W(v, \theta, t) < 1\}$.

Since $\Lambda \subseteq$ E(W), then the authors have $v \in B_W(v, r, t) \neq \emptyset$ for all $v \in X, r > 0$ and $t > 0$.

The collection is known as the neighborhood system for the topology τ_W and defined by $\beta = \{B_W(v, r, t) : v \in X, r > 0, t > 0\}$ on X induced by GRFM U_W. Openly, U is a subset of X that is known as open for each $v \in X$ if there exists an $r > 0$ such that $B_W(v, r, t) \subset U$ for $t > 0$. Indeed, if complement X/C is open of a subset C of X then it is known as closed.

Lemma 2.9.

Every open ball in X is an open set.

Proof.

Suppose an open ball $B_W(v, r, t)$ with center v, radius r, and time t. Assume that $\theta = B_W(v, r, t)$, then $U_W(v, \theta, t) < r$. Since $U_W(v, \theta, t) < r$, there exists $t_0 \in (0, t)$ such that $U_W(v, \theta, t_0) < r$.

Put $r_0 = U_W(v, \theta, t_0)$. As $r_0 < r$, there exists $u \in (0, t)$ such that $r_0 < u < r$. At this instant for specified r_0 and u such that $r_0 < u$, there exists $r_1, r_2 \in (0, t)$ such that $r_0 \oplus r_1 < u$. Set $r_3 = min\{r_1, r_2\}$ and let the open ball $B_W(\theta, r_3, t - t_0)$. We claim that $B_W(\theta, r_3, t - t_0) \subset B_W(v, r, t)$. By definition, there is $(vP\theta)_W$ and $(\theta Pu)_W$ such that $(vPw)_W$. At this instant, suppose $u \in B_W(\theta, r_3, t - t_0)$. Then $U_W(\theta, u, t - t_0)$. Therefore,

$U_W(v, u, t) \leq U_W(v, \theta, t_0) \oplus U_W(\theta, u, t - t_0) \leq r_0 \oplus r_3 \leq r_0 \oplus r_1 < u < n$.

Therefore $u \in B_W(v, r, t)$ and $B_W(\theta, r_3, t - t_0) \subset B_W(v, r, t)$.

Next, in GRFMS, we define the terms convergence, Cauchy sequence, and completeness.

Definition 2.10.

Assume $(X, U_W, \oplus)$ to be a GRFMS and a sequence $\{v_n\}$ is convergent and converges to $v \in X$ if, for known $\varepsilon > 0$, there exists $n_0 \in N$ such that $U_W(v_n, u, t) < \varepsilon$ for all $n > n_0$. It is obvious that the sequence $\{v_n\}$ is convergent and converges to v, if and only if $\lim_{n \to +\infty} U_W(v_n, u, t) = 0$.

Lemma 2.11.

Suppose $(X, U_W, \oplus)$ to be GRFMS with topology τ_W induced by GRFM. Then τ_W is a T_1 but not T_2 generally, i.e., Hausdorff.

Proof.

We will demonstrate that for each $v \in X$, the set $\{v\}$ that is singleton is a closed subset of X, i.e., there set $X\backslash\{v\}$ is open in X. Assume $\theta \in X\backslash\{v\}$, then $\theta \neq v$, i.e., $U_W(v,\theta,t) < 1$.

Now, let $r = \frac{U_W(v,\theta,t)}{2} < 1$. Then we observe that $v \neq B_W(\theta,r,t)$, otherwise $\frac{U_W(v,\theta,t)}{2} = r < U_W(v,\theta,t)$, which is a contraction. It shows that $B_W(\theta,r,t) \subset X\backslash\{v\}$, and so $X\backslash\{v\}$ is open.

Definition 2.12.

Let $(X,U_W,\oplus)$ be a GRFMS and a sequence $\{v_n\}$ is known as a Cauchy sequence if, specified $\varepsilon > 0$, thereexists $n_0 \in N$ such that $U_W(v_n,v_m,t) < \varepsilon$ for all $n,n > n_0$. It is obvious that the sequence $\{v_n\}$ is a Cauchy sequence if and only if $\lim\limits_{n,m\to+\infty} U_W(v_n,v_m,t) = 0$.

Definition 2.13.

A GRFMS $(X,U_W,\oplus)$ is known as complete if each Cauchy sequence in X converges in X. suppose that $W^\diamond$ is a different graph from W such that $V(W^\diamond) = X$, if $W^\diamond$ −termwise connectedthe sequence in X, then $(X,U_W,\oplus)$ is known as $W^\diamond$ −complete.

3. Fixed point theorems in GRFMS

In this section, we discuss the existence of fixed point theorems under the circumstances.

Definition 3.1.

Suppose $(X,U_W,\oplus)$ to be a GRFMS, a mapping $\Gamma: X \to X$ and $W^\diamond$ to be a sub-group of W such that $\Lambda \subseteq E(W^\diamond)$. then, Γ is known as $(W,W^\diamond)-$revised fuzzy graphical contraction $\left((W,W^\diamond) - R_{FGC}\right)$ on X if satisfying the following conditions:

(R_{FGC} 1) $(\mathrm{v},\theta) \in E(W^\diamond)$ implies $(\Gamma\mathrm{v},\Gamma\theta) \in E(W^\diamond)$ at a time, that is, Γ preserves edges in $E(W^\diamond)$.

(R_{FGC} 2) There exists $\omega \in (0,1)$ such that:

$U_W(\Gamma\mathrm{v},\Gamma\theta,\omega t) \leq U_W(\mathrm{v},\theta,t)$, for all $v,\theta \in X$ with $(\mathrm{v},\theta) \in E(W^\diamond)$ at time t.

A sequence $\{v_n\}$ with initial value $v_0 \in \Gamma$ is known as the $\Gamma-$picard sequence if $v_n = v_{n-1}$ for all $n \in N$. In order to continue the discussion, we support that $W^\diamond$ is a sug-graph of W such that $\Lambda \subseteq E(W^\diamond)$.

Theorem 3.2.

Suppose $W^{\diamond}$ −complete GRFMS $(X, U_W, \oplus)$, and let $\Gamma: X \to X$ be a $(W, W^{\diamond}) - R_{FGC}$. assume that the below circumstances fulfill:

a. $\Gamma \mathrm{v}_0 \in [\mathrm{v}_0]^{\mathrm{l}}_{W^{\diamond}}$ if there exists $v_0 \in X$ for some $l \in N$.

b. If there exists $\{v_n\}$, that is, $W^{\diamond}$ −termwise connected Γ −picard converges in X, then for $\{v_n\}$ there exists a limit $\omega \in X$ and $n_0 \in N$ such that $(v_n, \omega) \in E(W^{\diamond})$ or $(\omega, v_n) \in E(W^{\diamond})$ at a time t for all $n \geq n_0$.

Then there exists $v^{\oplus} \in X$ such that a sequence $\{v_n\}$ is a Γ −picard is $W^{\diamond}$ −termwise connected with initial value $v_0 \in X$ and converges to both $v^{\oplus}$ and $\Gamma v^{\oplus}$.

Proof.

Suppose $v_0 \in X$ to be the extent that $\Gamma \mathrm{v}_0 \in [\mathrm{v}_0]^{\mathrm{l}}_{W^{\diamond}}$ for some and the Γ −Picard sequence $\{v_n\}$ starting from v_0. After that, there exists a path $[\theta_{\mathrm{i}}]^{\mathrm{l}}_{i=0}$ such that $\mathrm{v}_0 = \theta_0$, $\Gamma \mathrm{v}_0 = \Gamma \theta_l$ and $(\theta_{l-1}, \theta_i) \in E(W^{\diamond})$ at a time for $i = 1, 2, \cdots, l$. Since Γ is a $(W, W^{\diamond}) - RFGC$, by (RFC 2), $(\Gamma \theta_{l-1}, \Gamma \theta_i) \in E(W^{\diamond})$ at a time t for $i = 1, 2, \cdots, l$.Therefore, $[\Gamma \theta_{\mathrm{i}}]^{\mathrm{l}}_{i=0}$ is a path from $\Gamma \theta_0 = \Gamma v_0 = v_1$ to $\Gamma \theta_l = \Gamma^2 v_0 = v_2$ of length l and so $v_2 \in [\mathrm{v}_1]^{\mathrm{l}}_{W^1}$. Continuing this process, we obtain that $[\Gamma^{\mathrm{n}} \theta_{\mathrm{i}}]^{\mathrm{l}}_{i=0}$ is a path from $\Gamma^{\mathrm{n}} \theta_0 = \Gamma^{\mathrm{n}} v_0 = v_n$ to $\Gamma^{\mathrm{n}} \theta_l = \Gamma^{\mathrm{n}} v_0 = v_{n+1}$ of length l and so, $v_{n+1} \in [\mathrm{v}_{\mathrm{n}}]^{\mathrm{l}}_{W^{\diamond}}$, for all $n \in N$. Thus $\{v_n\}$ is a $W^{\diamond}$-term-wise connected sequence. Since $(\Gamma^{\mathrm{n}} \theta_l, \Gamma^{\mathrm{n}} \theta_{l-1}) \in E(W^{\diamond})$ at a time t for $i = 1,2, \ldots, l$ and $n \in N$, by (R_{FGC} 2), we obtain:

$$U_W(\Gamma^{\mathrm{n}} \theta_{l-1}, \Gamma^{\mathrm{n}} \theta_l, \omega t) \leq U_W(\Gamma^{\mathrm{n}-1} \theta_{l-1}, \Gamma^{\mathrm{n}-1} \theta_l, \omega t) \leq \cdots \leq U_W\left(\theta_{l-1}, \theta_i, \frac{t}{\omega^n}\right) \tag{1}$$

Since $W^{\diamond}$ is a gub-graph of W and a $W^{\diamond}$ −term-wise connected sequence that is $\{v_n\}$, for any $n \in N$, we achieve by utilizing (d) of Definition 2.1, and (1), where we have:

$$U_W(v_n, v_{n+1}, t) = U_W(\Gamma^{\mathrm{n}} v_0, \Gamma^{\mathrm{n}+1} v_0, t) = U_W(\Gamma^{\mathrm{n}} \theta_0, \Gamma^{\mathrm{n}} \theta_l, t)$$
$$\leq \oplus_{i=1}^{l} [U_W(\Gamma^{\mathrm{n}} \theta_{l-1}, \Gamma^{\mathrm{n}} \theta_l, \omega t)] \leq \oplus_{i=1}^{l} \left[U_W\left(\theta_{l-1}, \theta_i, \frac{t}{\omega^n}\right)\right].$$

Therefore, the sequence $\{v_n\}$ is $W^{\diamond}$ −term-wise connected for $n, m \in N$ with $m > n$. we examined that

$$U_W(v_n, v_m, t) \leq \oplus_{i=1}^{m-1} [U_W(v_i, v_{i+1}, t)] \leq \oplus_{i=1}^{l} \left[U_W\left(\theta_{l-1}, \theta_i, \frac{t}{\omega^n}\right)\right].$$

Since $\omega \in (0,1)$, we obtain $\lim\limits_{n,m \to ++\infty} U_W(v_n, v_m, t) = 0$. Therefore, $\{v_n\}$ is a Cauchy sequence in X. utilizing the fact that X have $W^{\diamond}$ − completeness, the sequence $\{v_n\}$ converges in X and utilizing circumstance (B) , there exist $v^{\oplus} \in X$, $n_0 \in N$ such that

$(v_n, v^{\oplus}) \in E(W^{\diamond})$ or $(v^{\oplus}, v_n) \in E(W^{\diamond})$ at a time t, for all $n > n_0$ and $\lim_{n \to ++\infty} U_W(v_n, v_m, t) = 0$. Therefore $\{v_n\}$ converges to $v^{\oplus} \in X$, At this instant, if $(v_n, v^{\oplus}) \in E(W^{\diamond})$ at a time t, for all $n > n_0$, applying (RFC 2), we deduce:

$$U_W(v_{n+1}, \Gamma v^{\oplus}, t) = U_W(\Gamma v_n, \Gamma v^{\oplus}, t) \leq U_W\left(v_n, v^{\oplus}, \frac{t}{\omega}\right)$$

for all $n > n_0$. Since $\lim_{n \to ++\infty} U_W(v_n, v^{\oplus}, t) = 0$, we deduce $\lim_{n \to ++\infty} U_W(v_{n+1}, \Gamma v^{\oplus}, t) = 0$. A similar result holds if $(v_n, v^{\oplus}) \in E(W^{\diamond})$ at a time t, for all $n > n_0$, and this gives us the sequence $\{v_n\}$ that converges to $v^{\oplus}$ and $\Gamma v^{\oplus}$.

Example 3.3.

Assume $X = \left\{\frac{1}{2^n} \cup \{0\} : n \in N\right\}$ and a graph $W = W^{\diamond}$ with $V(W) = X$ and $E(W) = \Lambda \cup \{(v, \theta) \in X^2 : \theta \leq v\}$ at a time t. suppose $a \oplus b = a + b - ab$ and describes a mapping $U_W : X^2 \times (0, +\infty) \to [0,1]$ by :

$$U_W(v, \theta, t) = \begin{cases} 0 \; if \; v = \theta \\ \frac{v\theta}{t + v\theta} \; \; if \; v \in v, \theta \in X \backslash \{0\}, v \neq \theta \\ \frac{1}{2} \; if \; otherwise \end{cases}$$

Then, U_W is a GRFM on X and $(X, U_W, \oplus)$ is a $W-$complete GRFMS. Define a mapping $\Gamma : X \to X$ by:

$$\Gamma \mathrm{v} = \begin{cases} \frac{\mathrm{v}}{6}, & \text{if } \mathrm{v} = 0 \\ \frac{1}{6}, & \text{if } v \in \frac{1}{2^n} : n \in N. \end{cases}$$

Observe that Γ is an RFGC where $\omega = \frac{1}{2}$ and $t = 1$. for each $v \in \left\{\frac{1}{2^n} : n \in N\right\}$, we have $(v, \Gamma \mathrm{v}) \in E(W)$ at t, that is $\Gamma \mathrm{v} \in [\mathrm{v}]_{W^{\diamond}}^{l}$. Also, observe that all conditions of theorem 3.2 and (B) hold, but Γ has no fixed point in X.

Definition 3.4.

Let $(X, U_W, \oplus)$ be a GRFMS and $\Gamma : X \to X$ be a map. Then, the five-tuple $(X, U_W, \oplus, W, \Gamma)$ is known to have property (2) if:

Whenever a $W^{\diamond}-$termwise connected $\Gamma-$Picard sequence $\{v_n\}$ two limits $v^{\oplus}$ and $\theta^{\oplus}$,

if $v^{\oplus} \in X$ and $\theta^{\oplus} \in \Gamma(\mathrm{X})$, then (2)

Theorem 3.5.

Suppose a $W^{\diamond}-$complete GRFMS $(X, U_W, \oplus)$, and let $\Gamma : X \to X$ be a $(W, W^{\diamond})-RFGC$, Assume that

a. $\Gamma v_0 \in [v_0]^{l}_{W^\diamond}$ for there exists $v_0 \in X$ for some $l \in N$;
b. If sequence $\{v_n\}$ that is $W^\diamond$ −termwise connected Γ −Picard converges in X, then for $\{v_n\}$ there exists a limit $\omega \in X$ and $n_0 \in X$ such that $(v_n, \omega) \in E(W^\diamond)$ or $(\omega, v_n) \in E(W^\diamond)$ at a time t for all $n \geq n_0$.

Then, there exists $v^{\oplus} \in X$ such that a sequence $\{v_n\}$ that is Γ −Picard converges is $W^\diamond$ −termwise connected with initial vale $v_0 \in X$ and converges to both $v^{\oplus}$ and $\Gamma v^{\oplus}$. Furthermore, if five-tuple $(X, U_W, \oplus, W, \Gamma)$ contains the property (2), then Γ has a fixed point in X.

Proof.

Applying theorem 3.2, a sequence $\{v_n\}$ converges to $v^{\oplus}$ and $\Gamma v^{\oplus}$. Since $v^{\oplus} \in X$ and $\Gamma v^{\oplus} \in \Gamma X$, consequently by applying (2) we have $v^{\oplus} = \Gamma v^{\oplus}$.

We symbolize the set of all fixed points of Γ by $Fix(\Gamma)$, and the perception that we utilize is $X_\Gamma = \{v \in X : (v, \Gamma v) \in E(W^\diamond)\ at\ time\ t\}$.

Theorem 3.6.

Suppose $(X, U_W, \oplus)$ to be a $W^\diamond$ − complete GRFMS, and let $\Gamma: X \to X$ be a $(W, W^\diamond) - R_{FGC}$. assume that all conditions of theorem 3.5 are satisfied; then Γ has a fixed point. Furthermore, if X_Γ is weakly connected (as a sub-graph of $W^\diamond$), then the fixed point of Γ is unique.

Proof.

It is easy to observe that the existence of a fixed point follows from theorem 3.5. Now, assume that X_Γ is weakly connected. Let $v^{\oplus}, \theta^{\oplus} \in Fix(\Gamma)$ be two distinct fixed points of Γ. Therefore, $E(W^\diamond)$ includes all the loops, that is, $Fix(\Gamma) \subseteq X_\Gamma$ and so $v^{\oplus}, \theta^{\oplus} \in X_\Gamma$. We must get $(v^{\oplus}\Gamma\theta^{\oplus})_{W^\diamond}$ or $(\theta^{\oplus}\Gamma v^{\oplus})_{W^\diamond}$ because X_Γ is a weakly connected graph. Assume that $(v^{\oplus}\Gamma\theta^{\oplus})_{W^\diamond}$ such that there exists a sequence $[\theta_i]^{l}_{i=0}$, $v_0 = v^{\oplus}$, $v_l = \theta^{\oplus}$ and $(v_l, v_{i+1}) \in E(W^\diamond)$ at a time t and for $i = 0,1, \dots, l-1$. Therefore, Γ is a $(W, W^\diamond) - R_{FGC}$, and by consecutively utilizing (R_{FGC} 1), we deduce $(\Gamma^n \theta_i, \Gamma^n v_{i+1}) \in E(W^\diamond)$ at a time t for $i = 1, 2, \cdots, l-1$, and for all $n \in N$. Therefore, by (R_{FGC} 2) we have;

$$U_W(\Gamma^n v_i, \Gamma^n v_{i+1}, \omega t) \leq U_W(\Gamma^{n-1} v_i, \Gamma^{n-1} \theta_{i+1}, t) \leq \cdots \leq U_W\left(v_i, v_{i+1}, \frac{t}{\omega^n}\right)$$

for $i = 1, 2, \cdots, l-1$, and for all $n \in N$. Now, by (d) of Definition 1, we get

$$U_W(\Gamma^n v^{\oplus}, \Gamma^n \theta^{\oplus}, \omega t) \leq \oplus_{i=1}^{l-1}[U_W(\Gamma^n v_i, \Gamma^n v_{i+1}, \omega t)] \leq \oplus_{i=1}^{l-1}\left[U_W\left(v_i, v_{i+1}, \frac{t}{\omega^n}\right)\right].$$

Since $v^{\oplus}, \theta^{\oplus} \in Fix(\Gamma)$, we have $\Gamma^{n} v^{\oplus} = v^{\oplus}, \Gamma^{n} \theta^{\oplus} = \theta^{\oplus}$; consequently, letting $n \to +\infty$, as a result of the preceding inequality, we have $U_W(v^{\oplus}, \theta^{\oplus}, t) = 0$, which implies that $v^{\oplus} = \theta^{\oplus}$. Hence, the fixed point of Γ is unique.

4. Applications

In this section, we utilize the fixed-point technique to find a solution of integral equations and nonlinear fractional differential equations.

First, we utilize a fixed-point technique to find a solution of integral equations on X. We examine that the existence that the existence of a lower or upper solution to an integral equation implies the solution of the integral equation under certain conditions.

Let $X = C([0,1], R)$ be the set of all real valued continuous functions on $[0,1]$.

Let, $B = \left\{ v \in X : 0 \leq \underset{u \in [0,1]}{inf} v(u) \ and \ v(u) \leq 1, u \in [0,1] \right\}$

and describe the graph W, $W^{\diamond}$ by $W = W^{\diamond}$, $V(W) = X$ and:

$$E(W) = \Lambda \cup \{(\mathrm{v}, \theta) \in \mathrm{X}^2 : \mathrm{v}, \theta \in \mathrm{B}, \mathrm{v(u)} \leq \theta(\mathrm{u}), \text{for all } \mathrm{u} \in [0,1]\}.$$

Now consider a GRFM defined by $U_W : X^2 \times (0, +\infty) \to [0,1]$ as :

$$U_W(\mathrm{v(u)}, \theta(\mathrm{u}), \mathrm{t}) = \begin{cases} & if \ v = \theta 0 \\ \underset{u \in [0,1]}{inf} \dfrac{\mathrm{v(u)}\theta(\mathrm{u})}{\mathrm{t} + \mathrm{v(u)}\theta(\mathrm{u})} & \text{if } \mathrm{v}, \theta \in \mathrm{B}, \mathrm{v} \neq \theta; \\ \dfrac{1}{2} \ if \ otherwise & \end{cases}$$

Then $(X, U_W, \oplus)$ is a $W^{\diamond}$ −complete GRFMS with $a \oplus b = a + b - ab$. We suppose the integral equations as :

$$\alpha(u) = \int_0^1 S(u,c) f\big(c, v(c)\big) dc, \tag{3}$$

Where $f : [0,1] \times R \to R$ and $S : [0,1]^2 \to [0, +\infty)$ are continuous functions.

The function $\alpha(u) = \int_0^1 S(u,c) f\big(c, \alpha(c)\big) dc$, $\mathrm{u} \in [0,1]$.

It is obvious that reverse inequality is met by an lower solution of (3). Now, we examine the existence of a upper solution of (3) that reveals the existence of a solution of (3). We assume that $\Gamma : X^2$ is an operator and its described by

$$\Gamma(v)(u) = \int_0^1 S(u,c) f\big(c, v(c)\big) dc, \tag{4}$$

Which provides the necessary conditions for the existence of a fixed point of (4) in X. obviously, such a fixed point is a solution of the integral equation (3).

Theorem 4.1.

Assume that the requirement listed below are met:

a. $f(c,-): R \to R$ is decreasing on $[0,1)$, for each $c \in [0,1]$. Further $S(,c)f(c,0) < 1$:

b. there exists $\omega \in (0,1)$ such that for $v, \theta \in X$ with $(v,\theta) \in E(W)$ at t, for each $c, l \in [0,1]$, we have

$$f\big(c, v(c)\big) f\big(l, \theta(l)\big) \le v(c)\theta(l)$$

and

$$\iint_0^1 S(u,c)S(u,l)dcdl \le \omega, \qquad for\ all\ u \in [0,1]$$

Then the existence of a lower solution of (3) in B demonstrates the existence of the solution of (3) in X.

Conclusion

Today, graph theory is crucial to understanding and resolving a variety of issues in research. Here, we introduced a new GRFMS method and provided instructive examples. We discussed using integral equations and nonlinear fractional differential equations conditions for the presence of a unique solution for contraction mapping.

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