CHARACTERISATION OF THE TIME SERIES DECOMPOSITION MODELS WHEN THE TREND-CYCLE COMPONENT IS QUADRATIC

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Abstract

This study is on determining the characteristics of the three models commonly used in descriptive time series analysis when the trending is curve quadratic. The purpose is to provide basis for choosing the most appropriate decomposition model for any study data. The procedure is based on the row, column, and overall averages and standard deviation of the Buys-Ballot table proposed by Iwueze and Nwogu (2004) and Iwueze and Ohakwe (2004). Results show that for the additive model, the joint plots of the column means and standard deviations against column (j) appear to move upwards, the column means are greater than standard deviations and hence, the coefficients of variation are less than 100% and decreased with increasing j. For the mixed model, the joint plots of the column mean and standard deviation against column (j) appear to follow the seasonal pattern, the column means are greater than the standard deviations and hence, the coefficients of variation are less than 100% and decreased with increasing j. And for the multiplicative model, the joint plots of the column mean and standard deviation against column (j) appear to follow the seasonal pattern, the column means show no particular pattern of variation with the standard deviations and hence, the coefficients of variation are less than 100% for some columns and more than 100% for others columns. Therefore, the study recommends the use of the column means and standard deviations to characterize the decomposition models when trend-cycle component is quadratic.

Keywords: *Decomposition models, Characterization, Buys-Ballot table,column means and standard deviation, quadratic trend*

1 Introduction

In the descriptive time series analysis, the models commonly used for the decomposition of the observed data are the

Additive Model:

$$X_t = T_t + S_t + + C_t + e_t \tag{1.1}$$

Multiplicative Model:

$$X_t = T_t \times S_t \times C_t \times e_t \tag{1.2}$$

and Mixed Model/Pseudo-Additive;

$$X_t = T_t \times S_t \times C_t + e_t \tag{1.3}$$

where for time $t = 1, 2, ..., n$ X_t is the observed series, T_t is the trend (long term direction), S_t is the seasonal effect (systematic, calendar related movements), C_t is the cyclical (long term oscillations or swings about the trend) and irregular (e_t) (unsystematic, short term fluctuations) (Kendal and Ord, 1990; Chatfield, 2004). If short period of time are involved, the cyclical component is superimposed into the trend (Chatfield, 2004) and the observed time series $(X_t, t = 1, 2, ..., n)$ can be decomposed into the trend-cycle component (M_t), seasonal component (S_t) and the irregular/residual component (e_t). Therefore, the decomposition models are restated as

Additive Model:

$$X_t = M_t + S_t + e_t \tag{1.4}$$

Multiplicative Model:

$$X_t = M_t \times S_t \times e_t \tag{1.5}$$

and Mixed Model Pseudo-Additive/;

$$X_t = M_t \times S_t + e_t \tag{1.6}$$

However, one of the greatest challenges in descriptive time series analysis is when to use any of the three models for analysis. The reliability of any result from any analysis depends on how well the appropriate model was chosen. Use of wrong model will certainly lead to erroneous decomposition of a study series.

Several of attempts have been made by scholars in the literature to address this problem. Chatfield (2004) proposed the use of the run sequence plot (time plot). In his opinion, the appropriate model is additive if the seasonal variation stays roughly the same size regardless of the mean level, and multiplicative if it increases in size in direct proportion to the mean level. No test for the choice was provided and the choice is between additive and multiplicative models. According to Puerto and Rivera (2001), the appropriate model is Additive if the coefficient of variations of seasonal quotients (CV(c)) is greater than the coefficient of variations of seasonal differences (CV (d)). No test for the choice was provided and the choice is limited to between additive and multiplicative models. In his opinion, Linde (2005) says that the differences between the Multiplicative and the Additive models are: (i) in the additive model, the seasonal variation is independent of the absolute level of the time series, but it takes approximately the same magnitude each year while in the multiplicative model, the seasonal variation takes the same relative magnitude each year. Here again, no test for the choice was provided and the choice is between additive and multiplicative models.

The use of the relationship between the seasonal means $\left(\overline{X}_{.j}\right)$ and the seasonal standard deviations $\left(\hat{\sigma}_{.j}\right)$ of the Buys-Ballot table to choose the appropriate model for decomposition was proposed by Iwueze *et al* (2011). In their views, the additive model is appropriate when the seasonal standard deviations show no appreciable increase or decrease relative to any increase or decrease in the seasonal means. On the other hand, the appropriate model is multiplicative when the seasonal standard deviations show appreciable increase/decrease relative to any increase /decrease in the seasonal means. For details of Buys-Ballot table/procedure, see Wei (1989), Iwueze and Nwogu (2004 and 2005), Iwueze and Ohakwe (2004). However, in the proposals by Iwueze *et al* (2011), no test was provided for choice of appropriate model and the emphasis is still on additive and multiplicative models.

In a framework for choice of model and detection of seasonal effect in time series, Iwueze and Nwogu (2014) showed that when the trend-cycle component is linear, the column variances are constant for the additive model, but contain the seasonal component for the multiplicative model. Thus, choice between additive and multiplicative models reduces to test for constant variance to identify the additive model. Therefore, they suggested that any of the tests for constant variance can be used to identify a series that admits the additive model. In this method, Iwueze and Nwogu (2014) attempted to solve the problem of providing test for choice of model but this is limited to choice between additive and multiplicative models when trend-cycle component is linear.

Thus, in all these methods outlined above, including the Buys-Ballot procedure, the emphasis has been on the choice between additive and multiplicative models, with little or no reference to the mixed model. Furthermore, only Iwueze and Nwogu (2014) proposed a test for choice of model but this only identifies the additive model (when the column variance is constant), but does not tell the analyst the alternative model when the variance is not constant. The implication of this is that when the test for constant variance says the appropriate model for a study series is not the additive model; an analyst still faces the challenge of distinguishing between multiplicative model and the mixed model.

The only attempt which considered the mixed model is the work of Nwogu, Iwueze, Dozie and Mbachu (2019). In it, Nwogu et al (2019) proposed a statistical test for choosing between mixed and multiplicative models when trend-cycle component is linear based on the column variances of the Buys-Ballot table. However, this work is limited to choice between mixed and multiplicative models when trend-cycle component is linear. For other trending curves, it is not clear whether this test stands or not.

From the foregoing, it is clear that no attempt has been made at providing bases for choice model among the three decomposition models when the trend-cycle component is not linear. Therefore, the aim of this study is to provide bases for choosing the most appropriate decomposition model for any study series with quadratic trend-cycle component using the characteristics of the three models. Specifically, the method adopted is the Buys-Ballot method. So, (i) the first step taken here is the review of the Buys-Ballot procedures for the three decomposition models. The characteristics on bases of which the decomposition models are described are the row, column and overall means and variances of the Buys-Ballot table. (ii) So in the next step the row, column and overall means and variances are derived for each of the three decomposition models (iii) The characteristics which are peculiar to each of these models are identified and described. And hence, (iii) procedures for identifying the most appropriate model for any study series are proposed. (v) the proposed procedures are illustrated with empirical examples.

2 Methodology

Following the ways of Iwueze and Nwogu (2004, 2005) and Iwueze and Ohakwe (2004), the row, column and overall means and variances obtained for the three decomposition models are shown in Tables 2.1 through 2.3.

As Tables 2.1 through 2.3 show, the row means and variances are for

Additive model:

$$\overline{X}_{i}.= a+b\left(\frac{s+1}{2}\right)+c\frac{(s+1)(2s+1)}{6}$$
$$+\left[b+cs(s+1)\right](i-1)+cs^{2}(i-1)^{2}+\overline{e}_{i}. \tag{2.1}$$

$$\sigma_{i}^{2}=\sigma^{2}+\frac{s(s+1)}{180}\left[15b^{2}+30(s+1)bc+(2s+1)(8s+11)c^{2}\right]$$
$$+\frac{1}{s-1}\left[\sum_{j=1}^{s}S_{j}^{2}+2\left(bC_{1}+cC_{2}\right)\right]$$
$$+\left\{\frac{s^{s}(s+1)}{3}\left[b+(s+1)c\right]+\frac{2cC_{1}}{s-1}\right\}(i-1)+\left(2cs\right)^{2}\left(i-1\right)^{2} \tag{2.2}$$

Thus, in the additive model the row means and variances are quadratic functions of the row ($i-1$). The intercept and the coefficient of ($i-1$) of the row means do not depend on the seasonal effects. The row variances are also quadratic functions of the row ($i-1$). However, the intercept and the coefficient of ($i-1$) depend on the seasonal effects through C_1, C_2 and $\sum_{j=1}^{s} S_j^2$. This means, the row standard deviations may turn out to be linear in ($i-1$).

The row variance in (2.2) is a quadratic function of the row ($i-1$) and can be written equivalently as

$$\sigma_{i}^{2}=\left[\phi_{0}+\phi_{1}(i-1)\right]^{2}$$
$$=\phi_{0}+2\phi_{0}\phi_{1}(i-1)+\phi_{1}^{2}(i-1)^{2} \tag{2.3}$$

Equating corresponding components in (2.2) and (2.3) yields the expressions for ϕ_0 and ϕ_1. Thus, the standard deviation is linear in $(i-1)$. That is,

$$\hat{\sigma}_{i}=\phi_{0}+\phi_{1}(i-1) \tag{2.4}$$

Hence, in the regression equations of the row means and standard deviations on the rows for the Additive model, while the mean is a quadratic function of $(i-1)$, the standard deviation is a linear function of $(i-1)$.

For the mixed model the row mean and standard deviation are respectively,

$$\overline{X}_{i}. = a + bC_1 + cC_2 + \lfloor b + 2cC_1 \rfloor (i-1) + cs^2 (i-1)^2 + \overline{e}_i . \qquad (2.5)$$

$$\begin{aligned}\sigma_i^2 = \sigma^2 &+ \begin{bmatrix} a^2\sigma_{Sj}^2 + b^2\sigma_{jSj}^2 + c^2\sigma_{j^2Sj}^2 + 2ab\,Cov(S_j, jS_j) \\ + 2ac\,Cov(S_j, j^2S_j) + 2bc\,Cov(jS_j, j^2S_j) \end{bmatrix} \\ &+ 2s\begin{bmatrix} ab\sigma_{Sj}^2 + 2bc\sigma_{jSj}^2 + (b + 2ac)Cov(S_j, jS_j) \\ + bc\,Cov(S_j, j^2S_j) + 2c^2\,Cov(jS_j, j^2S_j) \end{bmatrix}(i-1) \\ &+ s^2\begin{bmatrix} (b^2 + 2ac)\sigma_{Sj}^2 + 4c^2\sigma_{jSj}^2 + 6bc\,Cov(S_j, jS_j) \\ + 2c^2 s\,Cov(jS_j, j^2S_j) \end{bmatrix}(i-1)^2 \\ &+ 2s^2 \lfloor bcs\sigma_{Sj}^2 + 2c^2\,Cov(S_j, jS_j) \rfloor (i-1)^3 + (c^2 s^4 \sigma_{Sj}^2)(i-1)^4 \end{aligned} \qquad (2.6)$$

Thus, in the mixed model the row means are quadratic functions of the row $(i-1)$. The coefficient (cs^2) of $(i-1)^2$ in mixed model is the same as in the additive model. However, in contrast to the additive model, the intercept and the coefficient of $(i-1)$ depend on the seasonal effects through C_1 and C_2. The row variance is a fourth order polynomial (quartic) function of the row $(i-1)$ plus the error variance. All the coefficients of regression of row variances on $(i-1)$ depend on the sum of squares and cross-products of the seasonal effects. The row variance is a polynomial of order four in $i-1$ and as such, can be written in equivalent form as

$$\begin{aligned}\sigma_i^2 &= \left[\phi_0 + \phi_1 (i-1) + \phi_2 (i-1)^2\right]^2 \\ &= \phi_0 + 2\phi_0\phi_1 (i-1) + (\phi_1^2 + 2\phi_0\phi_2)(i-1)^2 \\ &\quad + 2\phi_1\phi_2 (i-1)^3 + \phi_2^2 (i-1)^4 \end{aligned} \qquad (2.7)$$

Equating corresponding components in (2.6) and (2.7) yields the expressions for ϕ_0, ϕ_1 and ϕ_2

Measure	Estimate
$\bar{X}_{i\cdot}$	$a+b\left(\frac{s+1}{2}\right)+c\frac{(s+1)(2s+1)}{6}+\left[b+cs(s+1)\right](i-1)+cs^2(i-1)^2+\bar{e}_{i\cdot}$
$\bar{X}_{\cdot j}$	$a+\frac{(n-s)b}{2}+\frac{(n-s)(2n-s)c}{6}+\left[b+c(n-s)\right]j+cj^2+S_j+\bar{e}_{\cdot j}$
$\bar{X}_{\cdot\cdot}$	$a+\frac{(n-s)b}{2}+\frac{(n-s)(2n-s)c}{6}+\left[b+c(n-s)\right]C_1+cC_2+\bar{e}_{\cdot\cdot}$
$\sigma^2_{i\cdot}$	$\sigma^2+\frac{s(s+1)}{180}\left[15b^2+30(s+1)bc+(2s+1)(8s+11)c^2\right]+\frac{1}{s-1}\left[\sum_{j=1}^{s}S_j^2+2\left[bC_1+cC_2\right]\right]$ $+\left\{\frac{s^s(s+1)}{3}\left[b+(s+1)c\right]+\frac{2cC_1}{s-1}\right\}(i-1)+4c^2s^2(i-1)^2$
$\sigma^2_{j\cdot}$	$\frac{n(n+s)}{180}\left\{15b^2+30(n-s)bc+(2n-s)(8n-11s)c^2+60\left[bc+(n-s)c^2\right]j+60c^2j^2\right\}+S_j^2$
$\sigma^2_{x\cdot}$	$\sigma_x^2=\sigma^2+\frac{n}{(n-1)}\left[\sum_{j=1}^{s}S_j^2+2bC_1+[+(n-s)\,C_1\,2\,C_2]c+\right]$ $+\frac{n(n+1)}{12}\left(b^2+2bc\right)$ $+\left\{\begin{matrix}\frac{n}{180(n-1)}\left[(n^2-s^2)(2n-s)(8n-11s)+(s^2-1)(2s+1)(8s+11)\right]\\+\frac{n(n-s)(s+1)}{36(n-1)}[(n+s)(6n+13s)+5(s-1)-(n-1)]\end{matrix}\right\}c^2$

Table 2.1: Row, Column and Overall Means and Variances for the Additive Model

Table 2.2: Row, Column and Overall Means and Variances for the Mixed Model

Measure	Estimate
$\overline{X}_{i}.$	$a + bC_1 + cC_2 + [b + 2cC_1](i-1) + cs^2(i-1)^2 + \bar{e}_i.$
$\overline{X}._{j}$	$\left\{a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} + [b + c(n-s)]j + cj^2\right\}S_j + \bar{e}._j$
$\overline{X}..$	$a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} + [b + c(n-s)]C_1 + cC_2 + \bar{e}..$
$\sigma^2_{i}.$	$\sigma^2 + [a^2\sigma^2_{Sj} + b^2\sigma^2_{jSj} + c^2\sigma^2_{j^2Sj} + 2ab\,Cov(S_j, jS_j) + 2ac\,Cov(S_j, j^2S_j) + 2bc\,Cov(jS_j, j$ $+ 2s[ab\sigma^2_{Sj} + 2bc\sigma^2_{jSj} + (b + 2ac)Cov(S_j, jS_j) + bc\,Cov(S_j, j^2S_j) + 2c^2\,Cov(jS_j, j^2S_j$ $+ s^2[(b^2 + 2ac)\sigma^2_{Sj} + 4c^2\sigma^2_{jSj} + 6bc\,Cov(S_j, jS_j) + 2c^2 s\,Cov(jS_j, j^2S_j)](i-1)^2$ $+ [s^2(2bcs\sigma^2_{Sj} + 4c^2\,Cov(S_j, jS_j))](i-1)^3 + (c^2 s^4 \sigma^2_{Sj})(i-1)^4$
$\sigma^2_{j}.$	$\frac{n(n+s)}{180}\{15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2 + 60[bc + (n-s)c^2]j + 60c^2 j^2\}S^2_j +$
$\sigma^2_{x}.$	$\frac{n}{n-1}\frac{(n^2-s^2)}{180}\{15(b + 2cC_1)^2 + 30(n-s)(b + 2cC_1)c + (2n-s)(8n-11s)c^2\}$ $\frac{n}{n-1}\left\{\left[a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6}\right]^2 + \frac{(n^2-s^2)}{12}[b + c(n-s)c]^2 + \frac{(n^2-s^2)}{180}[n^2 - ($ $+ \frac{n}{n-1}\left\{[b + (n-s)c]^2 + \frac{(n-s)^2 c^2}{3}\right\}\sigma^2_{jS_j} + \frac{n}{n-1}c^2\sigma^2_{j^2S_j}$ $\frac{n}{n-1}2\left[ab + \frac{(n-s)}{2}b^2 + (n-s)ac + \frac{(n-s)(2n-s)}{2}bc + \frac{n(n-s)^2}{2}c^2\right]Cov(S_j, jS_j)$ $\frac{n}{n-1}\left\{2\left[ac + \frac{(n-s)}{2}bc + \frac{(n-s)(2n-s)}{6}c^2\right]\right\}Cov(S_j, j^2S_j) +$ $\frac{n}{n-1}\{2[bc + (n-s)c^2]\}Cov(jS_j, j^2S_j) + \sigma^2$

Table 2.3: Row, Column and Overall Means and Variances for the Multiplicative Model

Measure	Estimate
$\bar{X}_{i}.$	$\frac{1}{s}\left\{\left[a+b(i-1)+cs^{2}(i-1)^{2}\right]\sum_{s}^{s}S_{j}e_{ij}+\left[b+2c(i-1)\right]\sum_{s}^{s}jS_{j}e_{ij}+c\sum_{s}^{s}j^{2}S_{j}e_{ij}\right\}$
$\bar{X}._{j}$	$\left\{\left[a+bj+cj^{2}\right]\bar{e}._{j}+s\left[b+2cj\right]\frac{1}{m}\sum_{i=1}^{m}(i-1)e_{ij}+cs^{2}\frac{1}{m}\sum_{i=1}^{m}(i-1)^{2}e_{ij}\right\}S_{j}$
$\bar{X}..$	$\bar{X}..=\frac{1}{n}\left[a\sum_{i=1}^{m}\sum_{j=1}^{s}S_{j}e_{ij}+\sum_{i=1}^{m}\left[bs(i-1)+cs^{2}(i-1)^{2}\right]\sum_{j=1}^{s}S_{j}e_{ij}+\sum_{i=1}^{m}\left[b+2cs(i-1)\right]\sum_{j=1}^{s}jS_{j}e_{ij}\right]$ $+\frac{c}{n}\sum_{i=1}^{m}\left[b+2cs(i-1)\right]\sum_{j=1}^{s}j^{2}S_{j}e_{ij}$
$\sigma_{i}^{2}.$	$\frac{\sigma^{2}}{s}\sum_{j=1}^{s}\left(a+2bj+2cj^{2}\right)^{2}S_{j}^{2}+\frac{\sigma^{2}}{s}\left[2s\sum_{j=1}^{s}\left(a+2bj+2cj^{2}\right)\left(b+4cj\right)S_{j}^{2}\right](i-1)$ $+\frac{\sigma^{2}}{s}\left[s^{2}\sum_{j=1}^{s}\left(2ac+b^{2}+12bcj+20c^{2}j^{2}\right)S_{j}^{2}\right](i-1)^{2}$ $+\frac{\sigma^{2}}{s}\left(2cs^{3}\sum_{j=1}^{s}\left(b+4cj\right)S_{j}^{2}\right)(i-1)^{3}+\frac{\sigma^{2}}{s}\left(c^{2}s^{4}\sum_{j=1}^{s}S_{j}^{2}\right)(i-1)^{4}$
$\sigma_{j}^{2}.$	$\frac{n(n+s)}{180}\left\{15b^{2}+30(n-s)bc+(2n-s)(8n-11s)c^{2}+60\left[bc+(n-s)c^{2}\right]j+60c^{2}j^{2}\right\}S_{j}^{2}\ \sigma$
$\sigma_{x}^{2}.$	$\frac{\sigma_{2}^{2}}{n}\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\left[a+bs(i-1)+cs^{2}(i-1)^{2}\right]+2cs(i-1)j+bj+cj^{2}\right\}^{2}S_{j}^{2}$

$$B_{j}=\frac{n(n+s)}{180}\left\{15b^{2}+30(n-s)bc+(2n-s)(8n-11s)c^{2}+60\left[bc+(n-s)c^{2}\right]j+60c^{2}j^{2}\right\}$$

Thus, the standard deviation is Quadratic in $(i-1)$. That is,

$$\hat{\sigma}_i = \phi_0 + \phi_1 (i-1) + \phi_2 (i-1)^2 \tag{2.8}$$

Hence, in the regression equations of the row means and standard deviations on the rows, both are quadratic functions of $(i-1)$.

The row mean and variances for the multiplicative model are respectively,

$$\overline{X}_{i\cdot} = \frac{1}{s}\left\{ \begin{aligned} &\sum_{j=1}^{s}\left(a+bj+cj^2\right)S_j e_{ij} + \left[\sum_{j=1}^{s}(b+2cj)S_j e_{ij}\right](i-1) \\ &\qquad + \left[cs^2 \sum_{s}^{s} S_j e_{ij}\right](i-1)^2 \end{aligned} \right\} \tag{2.9}$$

$$\begin{aligned} \sigma_i^2 = {} & \frac{\sigma^2}{s}\sum_{j=1}^{s}\left(a+2bj+2cj^2\right)^2 S_j^2 \\ & + \frac{\sigma^2}{s}\left[2s\sum_{j=1}^{s}\left(a+2bj+2cj^2\right)(b+4cj)S_j^2\right](i-1) \\ & + \frac{\sigma^2}{s}\left[s^2\sum_{j=1}^{s}\left(2ac+b^2+12bcj+20c^2j^2\right)S_j^2\right](i-1)^2 \\ & + \frac{\sigma^2}{s}\left(2cs^3\sum_{j=1}^{s}(b+4cj)S_j^2\right)(i-1)^3 + \frac{\sigma^2}{s}\left(c^2s^4\sum_{j=1}^{s}S_j^2\right)(i-1)^4 \end{aligned} \tag{2.10}$$

As in the mixed and additive models, the row mean in the multiplicative model is a quadratic function of the row $(i-1)$. However, in contrast to the additive and the mixed models, all the coefficients of regression of the row means on $(i-1)$ depend on the seasonal effects and the error terms. The row variance is a fourth order polynomial (quartic) function of the row $(i-1)$. This means that the row standard deviations are quadratic in $(i-1)$ as in the mixed model. However, unlike the mixed model, all the regression coefficients depend on the square of the seasonal effects and the error variance.

As with the mixed model, the row standard deviation is quadratic function of $i-1$. That is,

$$\hat{\sigma}_i = \phi_0 + \phi_1 (i-1) + \phi_2 (i-1)^2 \tag{2.11}$$

Hence, in the regression equations of the row mean and standard deviation on the rows, both are quadratic functions of $(i-1)$.

From the foregoing, it is clear that in the regression equations of the row means and standard deviations on the rows for the Additive Model, while the row means are quadratic functions of the row $(i-1)$, the standard deviation is a linear function of the row $(i-1)$. For the Mixed and Multiplicative models, both the row means and standard deviations are quadratic functions of the row $(i-1)$.

Also from Tables 2.1 through 2.3, the column means and variances for the additive model are respectively,

$$\begin{aligned}\overline{X}_{.j} = a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} \\ + \left[b + c(n-s)\right]j + cj^2 + S_j + \bar{e}_{.j}\end{aligned} \tag{2.12}$$

$$\begin{aligned}\sigma_j^2 = \sigma^2 + S_j^2 + \frac{n(n+s)}{180}\left[15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2\right] \\ + \frac{n(n+s)}{3}\left[bc + (n-s)c^2\right]j + \frac{n(n+s)}{3}c^2 j^2\end{aligned} \tag{2.13}$$

The column mean is a quadratic function of the column (j) with the seasonal effect, the error mean and terms of the trend parameters as part of the intercept. And. The other regression coefficients are in terms of the trend parameters only.

The column variance, on the other hand, is a quadratic function of the column (j). In this regression, the intercept is the sums of the squares of the seasonal effects, the error variance and functions of the squares of the trend parameters, while the other regression coefficients are in terms of functions of the squares of the trend parameters. The column variance in (2.12) can be written in equivalent form as

$$\sigma_j^2 = [\psi_0 + \psi_1 j]^2 = \psi_0 + 2\psi_0 \psi_1 j + \psi_1^2 j^2 \qquad (2.14)$$

Equating corresponding components (2.12) and (2.13) yields the expressions for ψ_0 and ψ_1. Thus, the standard deviation is linear in $(i-1)$. That is,

$$\hat{\sigma}_j = \psi_0 + \psi_1 j \qquad (2.15)$$

Hence, in the regression equations of the column means and standard deviations while the column mean is a quadratic function of j, the column standard deviations is a linear function of j.

For the mixed model, the column means and variance are respectively,

$$\overline{X}_j = \left\{ \begin{array}{c} a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} \\ +[b + c(n-s)]j + cj^2 \end{array} \right\} S_j + \bar{e}._j \qquad (2.16)$$

$$\hat{\sigma}_j^2 = \sigma^2 + \frac{n(n+s)}{180}\left\{15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2\right\} S_j^2$$

$$+ \frac{n(n+s)}{180}\left\{60[bc + (n-s)c^2]j + 60c^2 j^2\right\} S_j^2 \qquad (2.17)$$

Equation (2.17) can be written in equivalent form as

$$\hat{\sigma}_j^2 - \sigma^2 = B_j S_j^2, \qquad (2.18)$$

where

$$B_j = \frac{n(n+s)}{180}\left\{\begin{bmatrix} 15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2 \\ + 60\left[bc + (n-s)c^2\right]j + 60c^2 j^2 \end{bmatrix}\right\} \tag{2.19}$$

Here also, as in the mixed model, the column variance is a quadratic function of j. Therefore, the column standard deviation is linear in j. However, the coefficients of regression equations of both the column mean and standard deviation are adjusted by the seasonal effects. That is

$$\overline{X}_j = A_j S_j + \bar{e}._j \tag{2.20}$$

where

$$A_j = a + \frac{(n-s)b}{2} + \frac{(n-s)(2n-s)c}{6} + \left[b + c(n-s)\right]j + cj^2 \tag{2.21}$$

and

$$\sqrt{\left(\hat{\sigma}_j^2 - \hat{\sigma}^2\right)} = \left(\psi_0 + \psi_1 j\right)\hat{S}_j, \tag{2.22}$$

where

$$B_j = \left(\psi_0 + \psi_1 j\right)^2 \tag{2.23}$$

For the multiplicative model, the column mean and standard deviation are given as

$$\overline{X}_j = \left\{\left[a + bj + cj^2\right]\bar{e}._j + s\left[b + 2cj\right]\frac{1}{m}\sum_{i=1}^{m}(i-1)e_{ij} + cs^2\,\frac{1}{m}\sum_{i=1}^{m}(i-1)^2 e_{ij}\right\}S_j$$

$$= \left(a\bar{e}._j + \frac{bs}{m}\sum_{i=1}^{m}(i-1)e_{ij} + \frac{cs^2}{m}\sum_{i=1}^{m}(i-1)^2 e_{ij}\right)S_j$$

$$+ \left[\left(b\bar{e}._j + \frac{2cs}{m}\sum_{i=1}^{m}(i-1)e_{ij}\right)j + \left(c\bar{e}._j\right)j^2\right]S_j \tag{2.24}$$

$$\hat{\sigma}_j^2 = \frac{n(n+s)}{180}\left\{15b^2 + 30(n-s)bc + (2n-s)(8n-11s)c^2\right\}S_j^2\ \hat{\sigma}^2$$

$$+ \frac{n(n+s)}{3}\left\{\left[bc + (n-s)c^2\right]j + c^2 j^2\right\}S_j^2\ \hat{\sigma}^2$$

$$= B_j S_j^2 \sigma^2 \ , \qquad (2.25)$$

where B_j is as defined in (2.19).

In the multiplicative model, both the column means and variances are a quadratic functions of j. Hence, the column standard deviation is linear in j. That is,

$$\hat{\sigma} = \left(\psi_0 + \psi_1 j\right) \hat{\sigma} \hat{S}_j \qquad (2.26)$$

Thus, in the regression equations of the column mean and standard deviation on j, the column mean is a quadratic function of j while the column standard deviation is linear in j. However, while the regression coefficients of the column mean on j are adjusted by the seasonal effects only, the regression coefficients of the column standard deviation are adjusted by the seasonal effects and the error standard deviation.

From the foregoing, it is clear that in the regression equations of the column means and standard deviations on j, while the column mean is a quadratic function of j, the column standard deviations is a linear function of j in all the decomposition models. However, in the additive model, the seasonal effect and error variance formed part of the intercept only. In the mixed model, all the regression coefficients are adjusted by the seasonal effects. While in the multiplicative models the regression equations are adjusted by the seasonal effects and error variance.

3. Empirical Examples

The empirical examples consist of 100 simulations of 120 observations each are from (a) $X_t = a + bt + ct^2 + S_t + e_t$ with a = 5.00, b = -0.35 and c = 0.35, $e_t \sim N(0, \sigma_1^2)$ and S_j given in Table 4.1 for the Additive model

(b) $X_t = (a + bt + ct^2)S_t + e_t$, with a = 5.00, b = -0.35 and c = 0.35, $e_t \sim N(0, \sigma_1^2)$ and S_j given in Table 4 for the Mixed model and

(c) $X_t = (a + bt + ct^2)S_t \, e_t$, with a = 5.00, b = -0.35 and c = 0.35, $e_t \sim N(1, \sigma_2^2)$ and S_j given in Table 4.2 for the Multiplicative model.

Table 3.1: Seasonal (S_j) indices used in the simulation of the Additive series.

j	1	2	3	4	5	6	7	8	9	10	11	12
S_j	- 1.5	2.5	3.5	- 4.5	-1.5	2.5	3.5	- 4.5	-1.5	2.5	3.5	-4.5

Table 3.2: Seasonal (S_j) indices used in the simulation of the Mixed and Multiplicative series.

j	1	2	3	4	5	6	7	8	9	10	11	12
S_j	0.91	0.88	1.00	0.98	0.98	1.12	1.26	1.20	1.05	0.92	0.80	0.90

The simulated series were obtained using the MINITAB software. The row, column and overall means and standard deviations of the series arranged in Buys-Ballot table are shown in Appendix A for the Additive, Appendix B for the Mixed model and Appendix C for the Multiplicative model, while the corresponding graphs are shown in Figures 4.1 through 4.3. For want of space the result for only ten simulations are reported in this study.

Row means and Standard deviations of the models

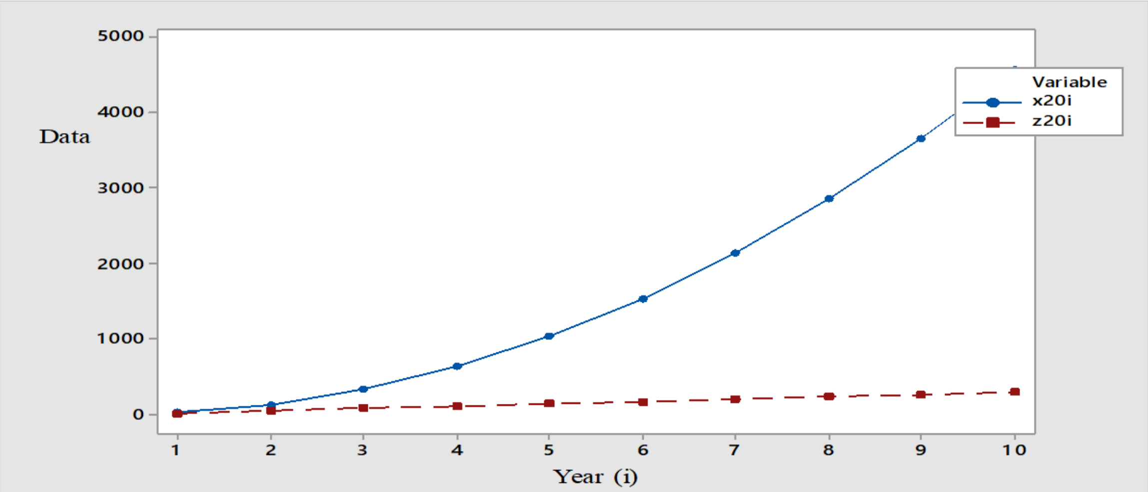


Fig. 4.1a: Time Plot of Row means (Xi) and Std Devs (Zi) - Add mod

As Figure 4.1a shows, for the additive model, the plot of the row means appears as a quadratic function of the row (i-1) while the plot of the row standard deviation appears as a

linear function of the row (i-1) as noted in Section 2. Similarly, Figure 4.1b shows that for the mixed model, the plot of the row means and standard deviation appears as a quadratic function of the row (i-1), though the curvature does not appear to be visible.

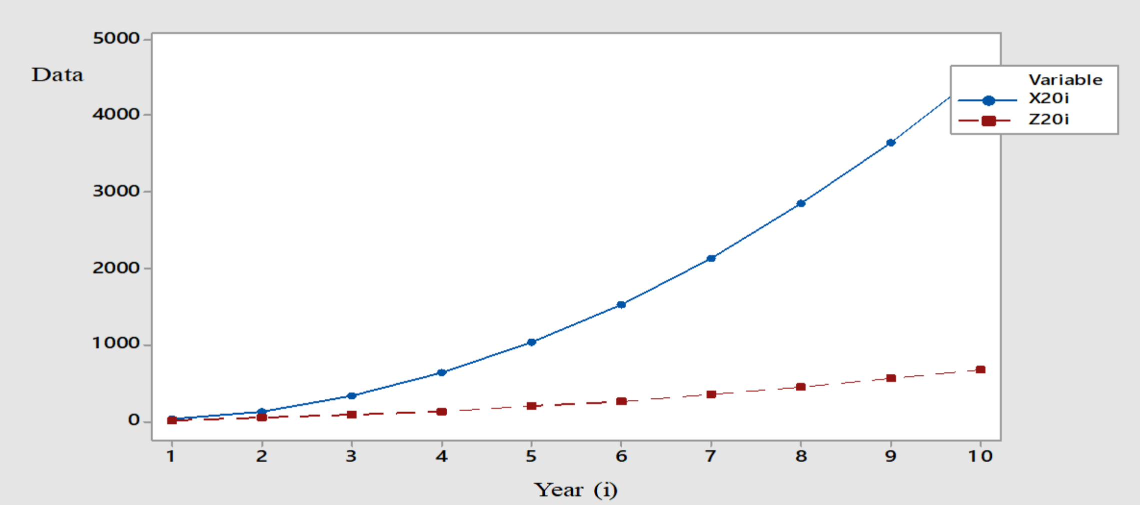


Fig. 4.1b: Time Plot of Row means (Xi) and Std Dev (Zi) - Mix Mod

For the multiplicative model, Figure 4.1c shows that the plot of the row means and standard deviation appear visibly as quadratic functions of the row (i-1).

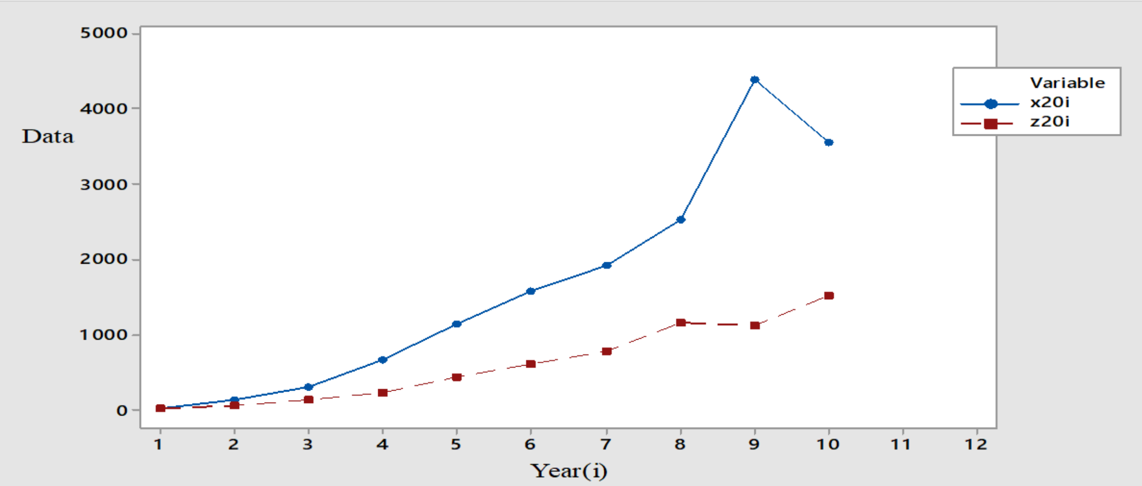


Fig. 4.1c: Time Plot of Row mean (Xi) and Std Dev (Zi)-Mult Mod

In all the decomposition models, the plots of the row means are clearly above the plots of the row standard deviations.

The column means and Standard deviations of the models

The joint plots of the column mean and standard deviations against j are shown in Figure 4.2. As Figure 4.2a shows, both plots appear to be growing in an upward direction form as the season (j) increased. In other words, the column means and standard deviations appear to be dominated by the trend. For all j, the column means remained higher than the standard deviations with the gap increasing as j increased. Recall that in the regression of the column means on j, the intercept is the sum of the trend parameters and the seasonal effects, while the other regression coefficients are in terms of the trend parameters only. The column standard deviation, on the other hand is a linear function of j. in the regression of the column standard deviation, the intercept contains the trend parameters, the seasonal effect and the error variance, while the other regression coefficients are in terms of the trend parameters only.

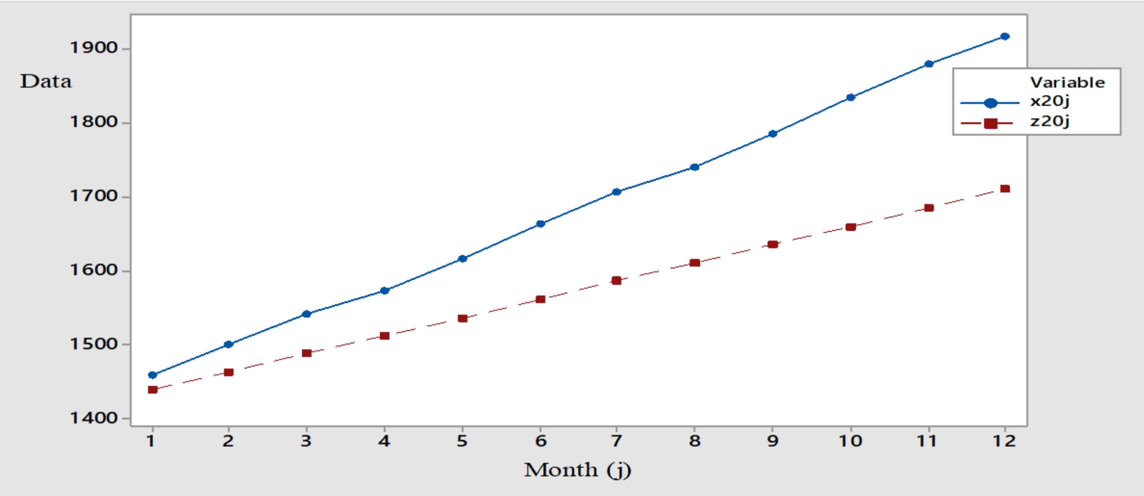


Fig. 4.2a: Time Plot of Column means (Xj) and Std Devs (Zj) - Add mod

For the mixed model, Figure 4.2b shows that the joint plots of the column mean and standard deviations against j varied according to the seasonal pattern. In other words, the column means and standard deviations appear to be dominated by the seasonal effects. Here again, the column mean remained consistently higher than the standard deviations, with the gap increasing with increasing j as in the Additive model. Recall for the mixed model, in the regression equations the column mean is a quadratic function of j while the column standard deviation is linear in j, thus putting the column mean above the column standard deviation. However, the coefficients of both regression equations are adjusted by the seasonal effects, producing curves that are S-shaped in upward direction.

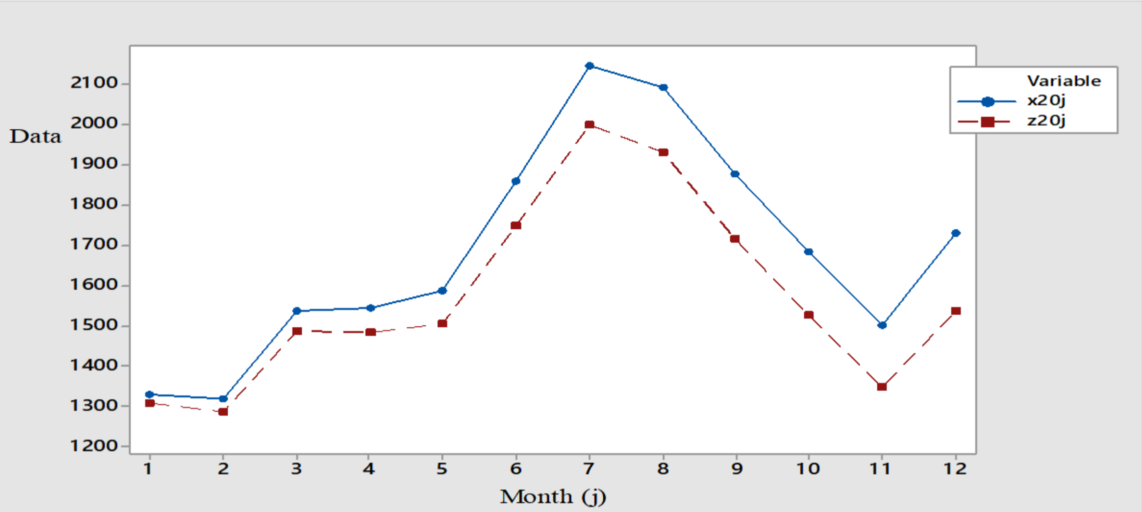


Fig. 4.2b: Time Plot of Column means (Xj) and Std Dve (Zj) - Mix Mod

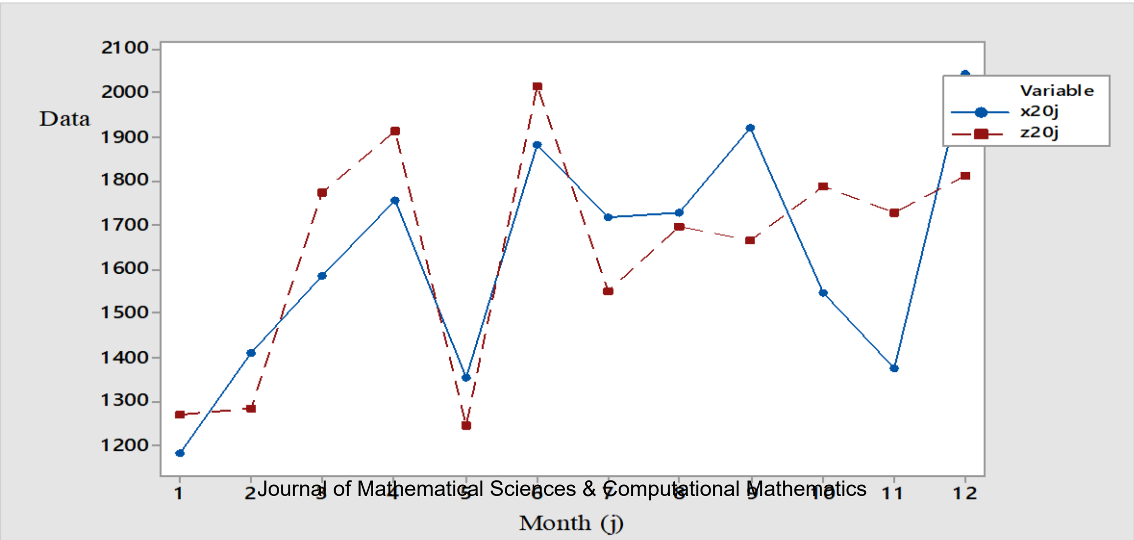
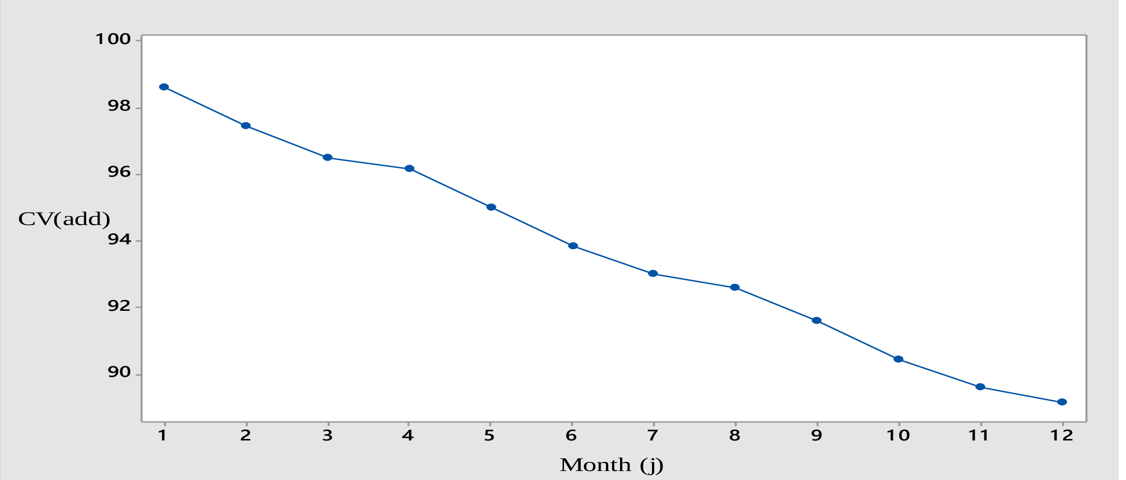


Fig. 4.2 c: Time Plot of Column means (Xj) and Std Dev (Zj) - Mult Mod

For the multiplicative model, Figure 4.2c shows that the joint plots of the column mean and standard deviations against j also varied according to the seasonal pattern as in the mixed model. In other words, the column means and standard deviations appear to be dominated by the seasonal effects. However, unlike the mixed model, the column mean is not consistently higher than the standard deviations, perhaps because of the additional influence of the error standard deviation. Recall, in the regression equations, the column mean of the multiplicative model is a quadratic function of j while the column standard deviation is linear in j, putting the column mean ahead of the column standard deviation. However, the coefficients of both regression equations are adjusted by the seasonal effects and error standard deviation producing curves that are S-shaped in upward direction.

Time plots of Column Coefficients of variation

From the relationships between the column mean and the column standard deviation, for the additive and mixed models, the column means are consistently higher than the column standard deviation. The implication of these is that Column Coefficients of Variation is for all j less than 100%. The widening gap between the column means and standard deviations implies decreasing Coefficients of Variation with increasing j as shown in Figures 4.3a and 4.3b.



ig 4.3a: Time Plot of coefficient of variation based on column means and standard deviation - add mo

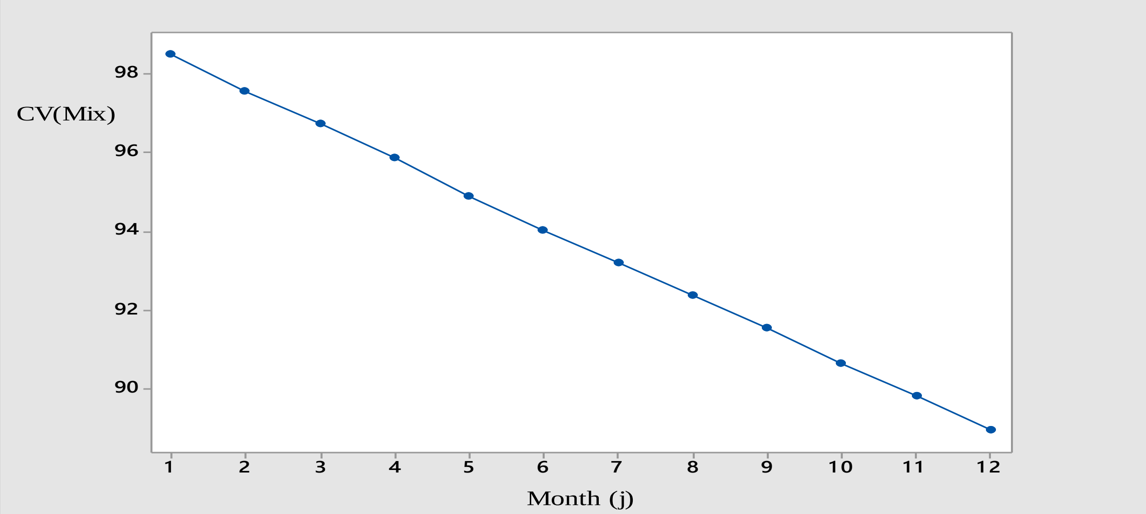


Fig 4.3c: Time Plot of Coefficient of Variation based on column means and std dev - Mix Mod

For the multiplicative model, the column means are not consistently higher than the column standard deviation. The implication of these is that Column Coefficients of Variation is for some j, less than 100% and for others greater than less than 100%, as shown in Figure 4.3c.

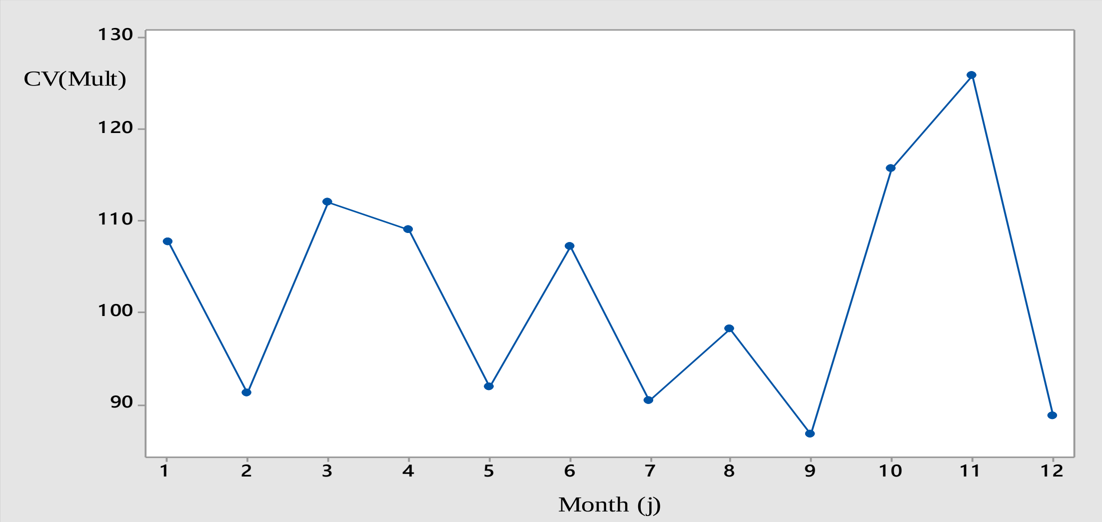


Fig 4.3b: Time Plot of coefficient of variation based on column means and std dev- Mult mod

4 Summary, Conclusion and Recommendation

This study discusses the characterization of decomposition models in descriptive time series when the trending curve is quadratic. The purpose is to provide bases for choosing the most appropriate model for decomposition of an observed series. This is necessary because, use of wrong decomposition leads to biased estimates of trend parameters and seasonal indices. Choice of appropriate model for time series decomposition has been one of the greatest challenges in use of descriptive time analysis. Available efforts at choice appropriate model for time series decomposition are limited to choice between additive and multiplicative models. The only effort that considered the mixed model is that of Nwogu at al (2019) which considered only the linear trending curve.

Buys-Ballot procedure proposed by Iwueze and Nwogu (2004 and 2014) and Iwueze and Ohakwe (2004) was adopted in this study. The row, column and overall means and standard deviations to derived from the Buys-Ballot were used to characterize the three decomposition models.

The results show that in all decomposition models except the additive, the row means and standard deviations are quadratic functions of the row $(i-1)$. In Additive model, while the plot of the row mean is a quadratic function of $(i-1)$, the plot of the row standard deviations is a linear function of $(i-1)$. In all decomposition models the plot of the row mean is consistently higher than the plot of the standard deviation. This is perhaps because for the additive model, the regression equation of the row mean on the row $(i-1)$ is a quadratic, while the regression equation of the row standard deviations on the row $(i-1)$

is a linear. For the mixed and multiplicative models, the regression equations of the row mean and standard deviation on $(i-1)$ are both quadratic. However, while the regression coefficients of the row mean depend on trend parameters and seasonal indices, the regression coefficients of the row standard deviations depend on the sum of squares and cross-products of the trend parameters and seasonal indices.

With regards to the column means and standard deviations, for the Additive model, both plots appear to be growing in an upward direction as the season (j) increased, perhaps because both the column means and standard deviations appear to be dominated by the trend. For all j, the column means remained higher than the standard deviations with the gap increasing as j increased, perhaps because while the regression equation of the column means on j is quadratic the regression equation of the column standard deviation on j is linear. This implies that for each j, the corresponding coefficient of variation is less than 100% which decreased with increasing j. For the mixed model, the column means remained higher than the standard deviations with increasing gap as j increased, perhaps because while the regression equation of the column means on j is quadratic, the regression equation of the column standard deviation on j is linear. This implies that the corresponding coefficients of variation is less than 100% which decreased with increasing j as in the additive model. However, unlike the additive model, the joint plot of the column means and standard deviations appear to follow the pattern of the seasonal effect. This may be because, in their regression equations, all the regression coefficients are adjusted by the seasonal effects. For the multiplicative model the joint plot of the column means and standard deviations appear to follow the pattern of the seasonal effect as in the mixed model. However, unlike the mixed model, the column means are not consistently higher than the standard deviations for all j, perhaps because in their regression equations, all the regression coefficients are adjusted by the seasonal effects and the error standard deviation.

In view of these observation, it is clear that the characterization of the three decomposition models may be based on the column means and standard deviations. For the additive model, the joint plots of the column means and standard deviations against column (j) appear to move upwards while the coefficients of variation remained less than 100% and decreased with increasing j. For the mixed model, the joint plots of the column means and standard deviations against column (j) appear to follow the seasonal pattern while the coefficients of variation remained less than 100% and decreased with increasing j. For the multiplicative model, the joint plots of the column means and standard deviations against column (j) appear to follow the seasonal pattern while the coefficients of variation shows no clear pattern of variation with j. While coefficient of variation remained less than 100% for some columns it is more than 100% for other columns

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Appendix A1: Row Means and Standard Deviations of Mixed Model

Row	X1i	Z1i	X2i	Z2i	X3i	Z3i	X4i	Z4i	X5i	Z5i	X6i	Z6i	X7i	Z7i	X8i	Z8i	X9i	Z9i	X10i	Z10i
1	21.65	13.77	21.13	13.54	21.3	13.49	21.07	13.81	20.99	14.13	21.42	14.33	21.47	14.38	21.31	14.27	21.33	14.39	21.24	13.52
2	121.9	44.6	121.7	43.8	121.8	43.4	121.9	43.8	122.1	44.1	121.8	43.2	121.7	43.8	122.0	43.8	121.9	44.5	122.0	43.8
3	323.2	81.6	323.7	81.2	323.3	81.4	323.7	81.9	322.9	81.5	323.1	81.5	323.5	81.6	323.4	81.3	323.7	81.3	323.4	81.5
4	625.5	128.6	626	128.9	626	128.2	626	128.6	625.8	128.9	625.7	128.5	625	128.3	626.1	128.4	625.8	129.1	625.4	128.6
5	1029.3	187.7	1029	187.5	1029	187.8	1029	187.4	1029	187.3	1029	186.7	1029	188	1029	187.7	1029	187.6	1030	187.6
6	1532.7	259	1533	258.3	1533	259	1533	259	1533	258.8	1533	259	1533	259.4	1533	258.8	1533	259.	1533	259.5
7	2137.6	343.8	2137	343.2	2138	343.7	2138	344.	2138	343.1	2138	343.5	2138	343.7	2138	343.2	2137	343.8	2138	344.
8	2843	441	2843	442.0	2843.	441	2843	441	2843	441.	2843	442.	2843	441	2843.	442	2843	441	2843.	441.
9	3649	553.	3649	552.0	3650	553	3649	552	3650	552.	3649	553	3650	553	3650	552.	3650.	552.	3650.	552.
10	4557.	677	4557	678	4557	678.	4557	678.	4557	677	4557	678	4557	677.	4557	678	4557	677	4556	678.
$\bar{X}$	1684	273.9	1684	272.8	1684	272.9	1684	272.9	1684	272.8	1684	272.9	1684	273	1648.	272.9	1684.	272.9	1684	272.9
STD	1574	225.9	1574	266.2	1574	226.3	1574	226	1574	225.7	1574	226.3	1574	225.9	1574	226.1	1574	225.7	1574	226.1

Appendix A2: Column Means and Standard Deviations of Mixed Model

Col	X1j	Z1j	X2j	Z2j	X3j	Z3j	X4j	Z4j	X5j	Z5j	X6j	Z6j	X7j	Z7j	X8j	Z8j	X9j	Z9j	X10j	X10j
1	1329	1310	1329	1309	1330	1310	1329	1309	1329	1310	1329	1309	1330	1309	1329	1309	1329	1309	1329	1309
2	1319	1288	1319	1287	1319	1287	1319	1288	1319	1288	1319	1287	1318	1288	1319	1287	1319	1288	1319	1287
3	1538	1488	1538	1488	1538	1488	1538	1488	1538	1488	1538	1487	1538	1489	1538	1488	1538	1488	1538	1488
4	1546	1482	1547	1482	1546	1482	1547	1482	1546	1482	1547	1483	1546	1482	1547	1482	1546	1483	1546	1482
5	1586	1506	1586	1506	1586	1506	1586	1507	1586	1507	1586	1506	1586	1506	1586	1507	1586	1507	1587	1506
6	1859	1749	1859	1749	1859	1750	1859	1749	1859	1749	1859	1749	1859	1749	1859	1750	1859	1749	1859	1749
7	2145	1999	2144	1999	2144	1998	2144	1999	2144	1999	2144	2000	2144	1999	2144	1999	2145	1998	2144	1999
8	2093	1933	2093	1934	2093	1934	2093	1933	2093	1933	2094	1933	2093	1933	2093	1933	2093	1933	2094	1933
9	18	17	18	171	18	17	187	17	18	17	18	17	18	17	18	17	18	17	18	17

	77	18	77	7	78	18	7	18	77	17	77	18	78	18	77	18	78	17	77	18
10	1686	1527	1685	1527	1685	1528	1686	1528	1685	1528	1685	1528	1685	1528	1686	1528	1685	1528	1685	1527
11	1502	1349	1501	1348	1501	1348	1502	1348	1502	1348	1501	1348	1501	1349	1502	1348	1501	1348	1501	1348
12	1730	1539	1730	1539	1730	1539	1730	1539	1730	1539	1329	1309	1730	1539	1730	1538	1730	1539	1730	1539
$\bar{X}$	1684	1574	1684	1573.8	1684.	1574	1684.2	1574	1684	1574	1685.	1600	1684	1574	1684	1574	1684	1574	1684	1574
STD	268.4	231	268.2	268.21	268.2	231.2	268.15	231.1	268.2	230.	268.4	230.3	268.3	231.	268.1	231.3	268.5	230.9	268.4	230.3

APPENDIX B1: ROW MEANS AND STANDARD DEVIATIONS OF THE ADDITIVE MODEL

Row	X1i	Z1i	X2i	Z2i	X3i	Z3i	X4i	Z4i	X5i	Z5i	X6i	Z6i	X7i	Z7i	X8i	Z8i	X9i	Z9i	X10i	Z10i
1	12.55	2.33	22.06	15.26	21.53	15.28	21.71	15.14	21.48	15.43	21.83	15.85	21.88	16.02	21.71	15.98	21.74	16.04	21.65	15.11
2	15.40	2.48	122.4	45.9	122.2	45.10	122.3	44.9	122.4	45.1	122.2	44.6	122.2	45.1	122.5	45.2	122.4	45.7	122.4	45.1
3	19.85	2.77	323.7	75.90	324.3	75.3	323.9	75.4	324.3	76.2	323.7	75.6	324.1	75.7	324	75.6	324.3	75.4	323.9	75.8
4	24.32	4.15	626.2	105.6	626.6	106.1	626.8	105	626.9	106.2	626.4	105.6	626.	105.5	626.8	106	626.4	106	626.0	105.9
5	31.36	5.23	1030	135.9	1029.	136.1	1030.	136.	1030	135.8	1029.	135.2	1030	136.1	1029	136.2	1030	136.1	1030.5	136.3
6	38.62	6.12	1534	166.2	1534	166.1	1534	166.2	1534	166.5	1534	166.4	1534	166.2	1534	165.9	1534	166.1	1534.	166.1
7	49.59	7.46	2139	196.8	2138	196.5	2139	196.6	2139	196.9	2139	196.9	2139	196.9	2139	196.5	2138	196.4	2138.7	196.7
8	61.88	9.31	2844	226.9	2844	226.7	2844	226.4	2844	227.1	2844	226.8	2844	226.4	2844	227.1	2844	227	2844.1	226.6
9	79.02	11.93	3650	257.3	3650.	256.9	3651	257.5	3650	256.8	3650	257.2	3651	257.3	3651	256.8	3651	257.2	3650.7	257.4
10	99.49	15.16	4558.	287.7	4558	287.4	4558	287.5	4558.	287.5	4558	287.6	4558	287.6	4558	287.8	4558	286.9	4557.4	287.8
$\bar{\bar{X}}$	42.21	6.70	1685	151.3	1685	151.1	1685	151.1	1685	151.4	1685	151.2	1685	151.3	1685	151.3	1685	151.3	1684.9	151.3
STD	29.09	4.32	1574	91.6	1574	91.6	1574	91.7	1574	91.5	1574	91.7	1574	91.5	1574	91.5	1574	91.3	1574.0	91.7

APPENDIX B2: COLUMN MEANS AND STANDARD DEVIATIONS OF THE ADDITIVE MODEL

col	X1j	Z1j	X2j	Z2j	X3j	Z3j	X4j	Z4j	X5j	Z5j	X6j	Z6j	X7j	Z7j	X8j	Z8j	X9j	Z9j	X10j	Z10j
1	1459	1440	1459	1439	1458	1439	1460	1439	1459	1438	1459	1439	1460	1439	1459	1438	1459	1439	1459	1439
2	1502	1463	1501	1464	1502	1463	1501	1462	1501	1464	1501	1463	1501	1463	1502	1463	1502	1464	1502	1463
3	1542	1488	1541	1488	1542	1488	1542	1488	1541	1488	1542	1487	1541	1489	1541	1488	1541	1488	1542	1488
4	1573	1512	1573	1513	1574	1512	1573	1512	1574	1513	1574	1513	1573	1512	1574	1512	1573	1513	1573	1512
5	1617	1537	1617	1537	1617	1537	1617	1537	1617	1537	1617	1537	1617	1537	1617	1538	1617	1538	1617	1537
6	1662	1562	1663	1562	1662	1562	1662	1562	1662	1562	1662	1561	1663	1562	1662	1562	1662	1562	1662	1562
7	1705	1586	1706	1586	1705	1586	1705	1586	1705	1587	1705	1587	1705	1586	1705	1587	1706	1586	1705	1587
8	1740	1611	1740	1611	1740	1612	1740	1611	1740	1610	1740	1611	1740	1611	1740	1611	1740	1611	1740	1611
9	1786	1636	1786	1636	1786	1635	1787	1636	1787	1636	1786	1636	1787	1636	1786	1636	1787	1635	1786	1636
10	1834	1661	1835	1660	1834	1660	1834	1661	1835	1660	1834	1661	1834	1661	1835	1660	1834	1661	1834	1660
11	1881	1685	1880	1686	1880	1685	1880	1685	1880	1685	1880	1685	1880	1686	1881	1685	1880	1685	1880	1685
12	1918	1710	1918	1710	1918	1710	1918	1710	1918	1710	1918	1710	1918	1710	1918	1709	1918	1710	1918	1710
$\bar{X}$	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574	1685	1574
Std	151.1	88.8	151.3	88.8	151.1	88.8	151.1	89.1	151.3	88.8	151.1	89.0	151.1	89.0	151.2	88.8	151.2	88.7	151	88.9

APPENDIX C1: Row Means and Standard Deviations of the Multiplicative Model

Row	X1i	Z1i	X2i	Z2i	X3i	Z3i	X4i	Z4i	X5i	Z5i	X6i	Z6i	X7i	Z7i	X8i	Z8i	X9i	Z9i	X10i	Z10i
1	20.09	15.16	22.82	15.45	18.77	10.78	19.6	14.26	19.34	14.67	23.43	19.48	23.9	19.68	22.53	18.58	23.19	16.62	19.29	13.65
2	126.1	63.2	125.6	70.0	110	46.0	109.2	45.6	117.1	49.2	103.7	41.2	110	57.70	118.5	51.70	123.1	76.90	118.4	39.4
3	273.4	134.7	294.2	117.9	337.2	123.2	306.5	107.9	355.8	164.4	288.3	132.8	328.4	173.6	309.3	109.8	338.5	122.1	312.7	139.8
4	631.	234.	570.	236.8	662.	246.	658.	194.	699.	234.	598.	189.	532.	103	673.	165.	630.	249.	537.	164.

	1	5	9		5		5	6	3	8	7		6		1			7	7	3
5	1028.	249.9	1144	286.5	954.4	322.7	1124	304.1	1000.	434.0	868.8	241.3	1026.	402.0	937.6	326.1	1075.	356	1266	300.6
6	1613	493	1460	549	1516	541	1471	553	1478	558	1575	448	1481	546	1466	512.	1569	414	1708	598.
7	2031	785	2141	588	1971	539.	2231	685	2401	855	2279	679	2235	670	2170	566	2067	765.	2318	817.
8	2845	792	3110	742	2676	1078	2464	624	2878.	809.	3136	896	2621	955	2885	797.	3057	828	2775	797
9	3701	755	3357	1051	3501	705	3808	1606	3454	942.	3527	1245	4226	936	3996	1091	3722	1071	3793	1036
10	4952	1250	5161	1144	4660	1715	4646	1712	5004	1825	4684	2060	4490	1002	4653	1784	4530	1587	4309	1660
$\bar{\bar{X}}$	1722.	477.3	1739	480.07	1641	532.7	1684	584.7	1741	1708	1708.	595.2	1707	486.5	1723.	542.1	1714	548.6	1716	556.6
STD	1664	405.7	1693	403.27	1559	530.0	1590	614.7	1651	1630	1640	625.3	1646	391.4	1652	557.8	1591	506.4	1546	530.2

APPENDIX C2: Column Means and Standard Deviations of the Multiplicative Model

Col	X1j	Z1j	X2j	Z2j	X3j	Z3j	X4j	Z4j	X5j	Z5j	X6j	Z6j	X7j	Z7j	X8j	Z8j	X9j	Z9j	X10j	Z10j
1	1608	1688	1407	1504	1249	1226	1793	1938	1268	1425	1321	1201	1589	1546	1305	1178	1360	1425	1272	1224
2	1470	1354	1461	1380	1390	1319	961	1024	1329	1258	1213	1154	1096	1203	1295	1166	1511	1611	1294	1230
3	1938	2039	1290	1392	1647	1614	1509	1324	1330	1381	1480	1325	1708	1925	1529	1654	1503	1544	1596	1346
4	1546	1792	1483	1633	1745	1840	1481	1416	1871	1856	1925	2045	1308	1373	1602	1571	1668	1775	1304	1218
5	1615	1800	1554	1716	1648	1524	1415	1409	1838	2181	1800	1969	1683	1654	1771	1933	1866	2013	1811	1691
	1	2	2	2	17	15	21	23	1	1	1	1	1	18	2	2	2	2	1	1

6	946	285	205	144	11	78	35	74	866	764	410	369	967	16	035	334	022	200	790	802
7	2092	1947	2210	2166	1699	1942	2097	1700	2367	2508	2355	2846	2221	2068	2406	2678	1943	1620	2307	2143
8	1883	1804	2038	1973	2307	2865	2064	2391	1774	1700	2224	2141	1902	1763	1998	2087	1803	1600	2241	2067
9	1496	1388	1965	1748	1710	1653	2129	2260	2264	2496	1827	2077	2097	2025	1757	1700	2319	2292	1975	2158
10	1816	2003	1701	1405	1403	1246	1522	1491	1859	1672	1790	1730	1588	1784	1773	1667	1697	1767	1468	1294
11	1558	1320	1790	2048	1235	1071	1282	1117	1443	1158	1360	1223	1693	1821	1611	1634	1314	1333	1604	1795
12	1697	1443	1759	1924	1945	1776	1816	1875	1679	1535	1795	1833	1635	1559	1597	1243	1554	1194	1926	1784
$\bar{\bar{X}}$	1722	1739	1739	1739	1641	1638	1684	1693	1741	1745	1708	1743	1707	1711	1723	1738	1731	1698	1716	1646
STD	205.8	205.8	308.8	312.3	301.22	469.22	381.44	474.76	353.9	448.9	359.9	510.1	314.6	257.41	313.9	460.9	292.9	333.5	354.7	369.5