INVERSE SOURCE IDENTIFICATION ON PARABOLIC EQUATION WITH CAUCHY CONDITIONS AS A GENERALIZED PROBLEM OF MOMENTS

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Abstract

We consider the problem of finding a pair of functions $r(x,t)$and $w(x,t)$that satisfy the equation $w_t(x,t) = w_{xx}(x,t) + r(x,t)$ under Cauchy boundary conditions. We will see that an approximate solution can be found using the techniques of generalized inverse problem of moments and find dimensions for the error of the estimated solution.

Keywords: *generalized moment problem; integral equations; Parabolic equation, inverse source.*

INTRODUCTION

We want to find $r(x,t)$and $w(x,t)$such that

$$w_t(x,t) = w_{xx}(x,t) + r(x,t)$$

about a domain about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $\mathrm{E} = (\mathrm{a}_1, \mathrm{b}_1) \times (\mathrm{a}_2, \infty)$ under the Cauchy conditions using the problem generalized moments techniques.

Inverse source problems occur in many branches of engineering sciences, for example, heat conduction, reaction diffusion, pollutant detection, crack identification, geophysical prospecting and electromagnetic theory. These problems are typically ill-posed in the sense of Hadamard. The solution (if it exists) does not depend continuously on measured data. So, the numerical simulation is very difficult and some special regularization methods and stability estimates are required.

The inverse source problems have been investigated in many papers; for example in [1] are to develop a numerical method to solve an inverse source problem for parabolic equations and apply it to solve a nonlinear coeficient inverse problema, in [2] the first attempt is made to investigate the abnormal spatio-temporal (S-T) source identification for a class of DPSs.
In [3] the source identification problem with non-local boundary conditions for the time-dependent, one-dimensional Schrödinger equation is studied. Stability estimates are constructed for the solution of source identification problem. A first order of accuracy difference scheme is investigated for the numerical solution of this problem.
In [4] the aim is to get instantaneous information at fixed instant T on pollution term in Navier-Stokes system in which the initial condition is incomplete.
In [5] is considered a reaction{di_usion equation where the reaction term is given by a cubic function and we are interested in the numerical reconstruction of the time-independent part of the source term from measurements of the solution.
In [6] is devoted to solve an inverse problem for identifying the source term of a time-fractional nonhomogeneous diffusion equation with a fractional Laplacian in a non-local boundary.

The multi-term time-fractional diffusion equation is a useful tool in the modeling of complex systems. In [7] aims to identifying a time-dependent source term in a multi-term time-fractional diffusion equation from the boundary Cauchy data.

In [8] it is studied an inverse source problem of the bi-parabolic equation. Here, it applies a Modified Quasi Boundary Method to deal with the inverse source problem.

In [9] it is studied uniqueness of a solution for an inverse source problem arising in linear time-fractional diffusion equations with time-dependent coefficients.
Is considered source term in a separated form $h(t)f(x)$.

In [10 it is studied an inverse parabolic problem of identifying two source terms in heat equation with dynamic boundary conditions from a final time overdetermination data.

In [11] a radial basis function collocation method (RBFCM) is proposed for the numerical treatment of inverse space-wise dependent heat source problems. Multiquadric radial basis function is applied for spatial discretization whereas for temporal discretization Runge-Kutta method of order four is employed

The diffusion process from some internal source arising in engineering situations can be mathematically described by a parabolic system. In [12] is considered an inverse source problema for parabolic system using parametric approximations, where deep neural networks (DNNs) are used to approximate the solution of the inverse problem.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. The interest is not to compare with the existing methods, but to present a different method.

The generalized moments problem [13, 14, 15, 16, 17] is to find a function $f(x)$about a domain $\Omega \subset \mathrm{R}^{\mathrm{d}}$ that satisfies the sequence of equations

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i \epsilon N - - - - - - - - - - - - (1)$$

where N is the set of the natural numbers, $\left(\mathrm{g_i(x)}\right)$ is a given sequence of functions in $\mathrm{L}^2(\Omega)$ linearly independent known and the succession of real numbers μ_i are known data.
The problem of Hausdorff moments [15, 17], is to find a function $f(x)$en (a, b) such that

$$\mu_i = \int_a^b x^i f(x) dx \quad i \epsilon N - - - - - - - - -(2)$$

In this case $g_i(x) = x^i$ with i belonging to the set N .
If the integration interval is $(0, \infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty, \infty)$ we have the problem of Hamburger moments [15, 17].
The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [15, 16]. There are several methods to build regularized solutions. One of them is the truncated expansion method [16].
This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i = 1,2, \dots, n - - - - - - - - - - - (3)$$

where it is considered as approximate solution of $f(x)$ to

$$p_n(x) = \sum_{i=0}^{n} \lambda_i \, \phi_i(x)$$

and the functions $\phi_i(x)_{i=0,1,\dots,n}$ result of orthonormalize $\langle \mathrm{g_1}, \mathrm{g_2}, \dots, \mathrm{g_n} \rangle$ being λ_i the coefficients based on the data μ_i . In the subspace generated by $\langle \mathrm{g_1}, \mathrm{g_2}, \dots, \mathrm{g_n} \rangle$ the solution is stable. If n ϵN is chosen in an appropriate way then the solution of (3) it approaches the solution of the original problem (1).
In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution ([16] p.p. 19 - 30).

ARTICLE ORGANIZATION

We want to find $r(x, t)$ and $w(x, t)$such that

$$\mathrm{w_t}(x, t) = w_{xx}(x, t) + r(\mathrm{x}, \mathrm{t})$$

about a domain about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$ under the Cauchy conditions using the problem generalized moments techniques.

We will see that the problem can be solved in three steps.

In steps one and two we found an approximation for $w(x,t)$. Then in a third step an approximate solution is found for $r(x,t)$

The next section describes the first step.
The section that follows it is explained how the generalized moment problem is solved with the truncated expansion method and explains the second step where an approximation for $w(x,t)$ is found.
Finally the numerical examples and the conclusions.

APPROXIMATION OF $w(x,t)$.

You want to find $w(x,t)$ such that

$$w_t = w_{xx} + r(x,t) - - - - - - - - (4)$$

about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$

We take as an auxiliary function for the finite case

$$u(m,z,x,t) = e^{-(m+1)\left(\frac{x}{b1}\right)} e^{-(z+1)\left(\frac{t}{b2}\right)}$$

For the unbounded case

$$u(m,z,x,t) = e^{-(m+1)\left(\frac{x}{b1}\right)} e^{-(z+1)(t)}$$

In what follows we considerer the case with E bounded

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w)$$

As

$$udiv(F^*) = -ur(x,t)$$

we have to:

Moreover, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, then

$$\iint_E u\, div(F^*) dA = \iint_E div(uF^*) dA - \iint_E F^* . \nabla u dA - - - - - - - - - - (5)$$

where $\nabla u = (u_x, u_t)$

Besides that

$$\iint_E div(uF^*)dA = \iint_E (uw_x)_x - (uw)_t dA =$$

$$= \iint_E u\, div(F^*)dA + \iint_E (u_x w_x - u_t w)dA ------(6)$$

Then from (5) y (6):

$$\iint_E (u_x w_x - u_t w_t)dA = \iint_E F^* . \nabla u dA -------------- (7)$$

On the other hand, it can be proven, after several calculations that, integrating by parts con respecto a x and writing $\left(\frac{m+1}{b_1}\right) = \left(\frac{z+1}{b_2}\right)$:

$$\varphi_1(m) = \int_{a_1}^{b_1} w(x,a_2)u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, x, a_2\right)\, dx$$
$$- \int_{a_1}^{b_1} w(x,b_2)u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, x, b\right)\, dx$$
$$+ \int_{a_2}^{b_2} w(b_1,t)u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, b_1, t\right)\, dx$$
$$- \int_{a_2}^{b_2} w(a_1,t)u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, a_1, t\right)\, dx \; =$$
$$= \iint_E u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, x, t\right)(w_x - w_t)\, dA ---------- (8)$$

To solve this integral equation we take a base $\psi_i(m) = rm^i e^{-m} \quad i = 0,1,2,\dots,n$ of $L^2(E)$. Then we multiply both members of (8) by $\psi_i(m) = rm^i e^{-m}$ and we integrate with respect to m, we obtain

$$\iint_E H_i(x,t)(w_x - w_t)dA = \int_{a_2}^{b_2} \varphi_1(m)\psi_i(m)dm = \mu_i \quad i = 0,1,2,\dots,n ----(9)$$

where $H_i(x,t) = \int_{a_2}^{b_2} u\left(m, \left(\frac{m+1}{b_1}\right)b_2 - 1, x, t\right)\psi_i(m)dm$.

We can interpret (9) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation $p_{1n}(x,t)$ for $w_x - w_t$.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [17] and generalized in [19] and [20] to find an approximation $p_{1n}(x,t)$ of $w_x - w_t$ for the corresponding finite problem with $i = 0,1,2,\dots,n$, where n is the number of moments μ_i.
We consider the basis $\phi_i(x,t)$ $\quad i = 0,1,2,\dots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t)$ $\quad i = 0,1,2,\dots,n$.
We approximate the solution $w_x - w_t$ with [13] and generalized in [19] y [20]:

$$p_{1n}(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{where} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\, \mu_j \qquad i = 0,1,2,\dots,n$$

and the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) | \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) . \|\phi_i(x,t)\|^{-1} \quad 1 < i \le n \,; 1 \le j < i \,.$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0,1,\dots,n.$$

The proof of the following theorem is in [19] and [20].
In [20] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [18] the demonstration is made for the one-dimensional case.
This Theorem gives a measure about the accuracy of the approximation.

Theorem

Let $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2\big((a_1,b_1)\times(a_2,b_2)\big)$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n} \left| \int_{a_2}^{b_2}\int_{a_1}^{b_1} H_i(x,t) f(x,t)\,dx\,dt - \mu_i \right|^2 \le \varepsilon^2$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} \left((b_1-a_1)^2 f_x^2 + (b_2-a_2)^2 f_t^2\right) dx\,dt \le M^2 - - - - - - - - - - - - (10)$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f(x,t)|^2\,dtdx \leq min_i\left\{\|C^T C\|\varepsilon^2 + \frac{M^2}{8(n+1)^2};i=0,1,\dots,n\right\}$$

where C it is a triangular matrix with elements C_{ij}, $1 < i \leq n;\ 1 \leq j < i$
and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_{1n}(x,t)-f(x,t)|^2\,dxdt \leq \|CC^T\|\varepsilon^2 + \frac{M^2}{8(n+1)^2} - - - - - - - - - - - - (11)$$

If b_2 it is not finite then (10}) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}(xf_x^2 + tf_t^2)dx\,dt \leq M^2$$

And it must be fulfilled that $t^i f(x,t) \longrightarrow 0$ if $t \longrightarrow \infty$ $\forall i \epsilon N$.

So we have an equation in first order partial derivatives of the form

$$w_x(x,t) - w_t(x,t) = p_{1n}(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t) + A_2(x,t)w_t(x,t) = p_{1n}(x,t)$$

where $A_1(x,t) = 1$ and $A_2(x,t) = 1$.
It is resolved as in [18], that is, we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}K(m,z,x,t)w(x,t)dtdx = \varphi_2(m,z) - - - - - - - - (12)$$

With $K(m,z,x,t) = u(m,z,x,t)\left(-\left(\frac{m+1}{b_1}\right) + \left(\frac{z+1}{b_2}\right)\right)$ where now it is taken as an auxiliary function

$$u(m,z,x,t) = e^{-(m+1)\left(\frac{x}{b_1}\right)}e^{-(z+1)\left(\frac{t}{b_2}\right)}$$

and

$$\varphi_2(m,z) = -\int_{a_1}^{b_1}(u(m,z,x,b_2)w(x,b_2) - u(m,z,x,a_2)w(x,a_2))dx -$$
$$-\int_{a_2}^{b_2}(u(m,z,b_1,t)\,w(b_1,t) - u(m,z,a_1,t)w(a_1,t))dt - \int_{a_2}^{b_2}\int_{a_1}^{b_1}p_{1n}(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,z) = m^i z^j e^{-(m+r)} \qquad i = 0,1,\dots,n_1 \quad j = 0,1,2,\dots,n_2$$

and we multiply both members of (12) by $\psi_{ij}(m,z)$ and we integrate with respect to m and z

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(x,t)H_{ij}(x,t) = \mu_{ij} ---------- (13)$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi_2(m,z)\psi_{ij}(m,z)dmdz$$

$$H_{ij}(x,t) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,z,x,t)\psi_{ij}(m,z)dmdz$$

We apply the truncated expansion method and find a numerical approximation $p_{2n}(x,t)$ for $w(x,t)$.

APPROXIMATION OF $r(x,t)$

We consider

$$w_t = w_{xx} + r(x,t) ---------- (14)$$

We take as an auxiliary function for the finite case

$$u(m,z,x,t) = e^{-(m+1)\left(\frac{x}{b1}\right)} e^{-(z+1)\left(\frac{t}{b2}\right)}$$

For the unbounded case

$$u(m,z,x,t) = e^{-(m+1)\left(\frac{x}{b1}\right)} e^{-(z+1)(t)}$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x, -w)$$

Since $div(F^*) = -r(x,t)$ we have to:

$$\iint_E u\, div(F^*)dA = -\iint_E u\, r(x,t)dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so

$$\iint_E u\, div(F^*)dA = \iint_E div(uF^*)dA - \iint_E F^*.\nabla u dA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_E F^*.\nabla u dA = \iint_E (F_1 u_x + F_2 u_t)dA$$

Integrating by parts:

$$\iint_E F_1\, u_x dA = \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_1\, u_x dxdt =$$

$$= \int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t)\big)dt - \iint_E wu_{xx}\, dA$$

then

$$\iint_D F^*.\nabla u dA =$$

$$= \int_{a_2}^{b_2}\big(w(b_1,t)u_x(m,z,b_1,t) - w(a_1,t)u_x(m,z,a_1,t)\big)dt - \iint_E wu\left(\left(\frac{m+1}{b_1}\right)^2 - \left(\frac{z+1}{b_2}\right)\right)dA$$

On the other hand,

$$\iint_E div(uF^*)dA = \int_C (uF^*).nds =$$

$$= -\int_{a_1}^{b_1} u(m,z,x,b_2)w(x,a_2)dx - \int_{a_1}^{b_1} u(m,z,x,a_2)w(x,b_2)dx +$$

$$+\int_{a_2}^{b_2} u(m,z,b_1,t)w_x(b_1,t)dt-\int_{a_2}^{b_2} u(m,z,a_1,t)w_x(a_1,t)dt=G(m,z)$$

$$\therefore \iint ur(x,t)dA=-G(m,r)+$$

$$+\int_{a_2}^{b_2}\left(w(b_1,t)u_x(m,z,b_1,t)-w(a_1,t)u_x(m,z,a_1,t)\right)dt+$$

$$+\iint_E wu\left(-\left(\frac{m+1}{b_1}\right)^2-\left(\frac{z+1}{b_2}\right)\right)dA=\varphi_3(m,z)----------(15)$$

we take a base:

$$\psi_{ij}(m,z)=m^i z^j e^{-(m+r)} \qquad i=0,1,\dots,n_1 \quad j=0,1,2,\dots,n_2$$

and we multiply both members of (15) by $\psi_{ij}(m,z)$ and we integrate with respect to m and z

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} r(x,t)H_{ij}(x,t)=\mu_{ij}----------(16)$$

where

$$\mu_{ij}=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\varphi_3(m,z)\psi_{ij}(m,z)dmdz$$

$$H_{ij}(x,t)=\int_{a_1}^{b_1}\int_{a_2}^{b_2}u(m,z,x,t)\psi_{ij}(m,z)dmdz$$

We apply the truncated expansion method and find a numerical approximation $p_{3n}(x,t)$ for $r(x,t)$.

NUMERICAL EXAMPLES

Example 1

We consider the equation

$$w_t(x,t)=w_{xx}(x,t)+r(x,t) \qquad E=(0,1)\times(0,2)$$

The solution is : $w(x,t)=\frac{e^{-t}}{(1+x)^2}$ $\quad r(x,t)=\frac{e^{-t}(7+2x+x^2)}{(1+x)^4}$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$, and $r(x,t)$ with $p_{3n}(x,t)$

To find $p_{1n}(x,t)$ we take $n = 5$ moments $(m=0,\dots,4)$
To find $p_{2n}(x,t)$ we take $n = 9$, moments $n_1=3, n_2=3$, in total $n=n_1.n_2=9$ moments.

$$\sqrt{\int_0^2\int_0^1\big(w(x,t)-p_{2n}(x,t)\big)^2\,dxdt}=0.0290012$$

In the Figure 1 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x,t)$ we take $n = 6$, moments $n_1=2, n_2=3$, in total $n=n_1.n_2=6$ moments.

$$\sqrt{\int_0^2\int_0^1\big(r(x,t)-p_{3n}(x,t)\big)^2\,dxdt}=0.326725$$

In the Figure 2 we show the exact solution (light blue) and the approximate solution (magenta color).

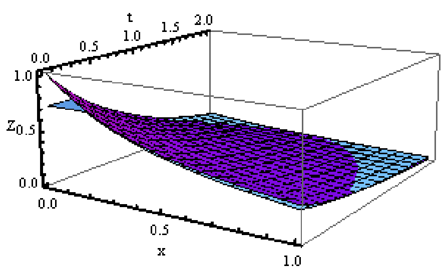

Fig. 1: $w(x,t)$ and $p_{2n}(x,t)$

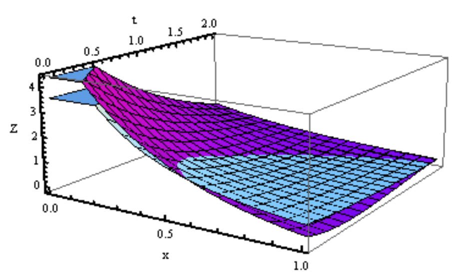

Fig. 2: $r(x,t)$ and $p_{3n}(x,t)$

Example 2

We consider the equation

$w_t(x,t)=w_{xx}(x,t)+r(x,t) \qquad E=(0,1)\times(0,\infty)$

The solution is

$$w(x,t) = \frac{e^{-t}}{(e^x + x)^2} \qquad r(x,t) = \frac{e^{-t}(2 + 2e^{2x} + x^2 + (4 + x)e^x)}{(e^x + x)^3}$$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$, and $r(x,t)$ with $p_{3n}(x,t)$

To find $p_{1n}(x,t)$ we take $n = 5$ moments ($m = 0, \ldots, 4$).
To find $p_{2n}(x,t)$ we take $n_1 = 3; n_2 = 3$, in total $n = n_1 . n_2 = 9$ moments.

$$\sqrt{\int_0^{\infty} \int_0^1 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.0253249$$

In the Figure 3 we show the exact solution (light blue) and the approximate solution (magenta color).

Analogously to find $p_{3n}(x,t)$ we take $n_1 = 2; n_2 = 3$, in total $n = n_1 . n_2 = 6$ moments.

$$\sqrt{\int_0^{\infty} \int_0^1 \left(r(x,t) - p_{3n}(x,t)\right)^2 dxdt} = 0.391679$$

In the Figure 4 we show the exact solution (light blue) and the approximate solution (magenta color).

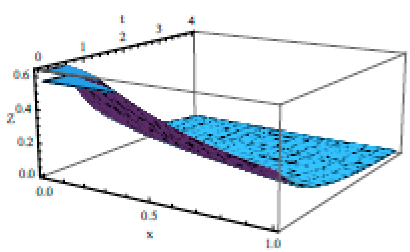

Fig. 3: $w(x,t)$ and $p_{2n}(x,t)$

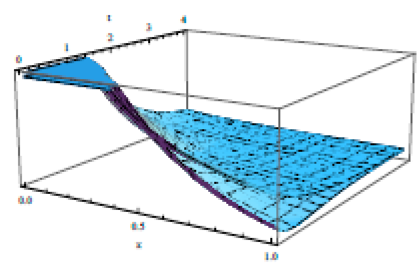

Fig. 4: $r(x,t)$ and $p_{3n}(x,t)$

CONCLUSION

The problem of finding $w(x,t)$ and $r(x,t)$ in the Parabolic equation $w_t = w_{xx} + r(x,t)$ about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$, under the Cauchy conditions it is possible to solve it using the problem generalized moments techniques.
First we can deduce $w(x,t)$ in two steps.

We consider the equation $w_t(x,t) = w_{xx}(x,t) + r(x,t)$, we turn it into an integral equation $\varphi_1(m) = \iint_E \ e^{-\left(\frac{m+1}{b_1}\right)x} e^{-\left(\frac{z+1}{b_2}\right)t}(w_x - w_t)\ dA$ and with the techniques of inverse problem moments we find an approximate solution $p_{1n}(x,t)$ para $w_x - w_t$.
Then we consider the equation $w_x(x,t) - w_t(x,t) = p_{1n}(x,t)$ and again we take it to an integral equation to solve it and find an approximate solution $p_{2n}(x,t)$ for $w(x,t)$..
Finally we consider $w_t(x,t) = w_{xx}(x,t) + r(x,t)$, and again we take it to an integral equation $\iint_E \ ur(x,t)dA = -G(m,z) + A(m,z) + \iint_E \ up_{1n}(x,t)\left(\left(\frac{m+1}{b_1}\right)^2 - \left(\frac{z+1}{b_2}\right)\right)dA = \varphi_3(m,z)$ and again we take it to an integral equation and find an approximate solution $p_{3n}(x,t)$ for $r(x,t)$. In the examples given it can be seen that due to their accuracy, the numerical approximations found for $w(x,t)$ and $r(x,t)$ are good.

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