FUZZY SET ON PERMUTATIONS OF Sn

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Abstract:
This study extends the symmetric group S_n (a symmetric group of n objects) to a fuzzy set and investigate various fuzzy properties. A constructed membership function was design to assign membership values ranging from 0 to 1, facilitating the examination of fuzzy characteristics. The alpha level set, support, height, and core of the fuzzy symmetric group were being investigated, providing insights into the fuzzy structure. The extension of S_n to a fuzzy set allows the capture of uncertainties and ambiguities inherent in permutation-based problems. The findings of this research contribute to the development of fuzzy group theory and its applications in algebra, The constructed membership function offers a simple way for assigning membership values, allowing for exploration of fuzzy properties. This work serves as an insight on fuzzy set in permutation and also allows further research in fuzzy group theory and its potential applications in various fields.

Keywords: *fuzzy set, alpha cut level, support, core, permutation*.

1. Introduction

The concept of fuzzy sets and its properties were first introduced by Zadeh (1965). Subsequently, many researchers such as Chang (1968), Lowen (1976), Hutton (1980) and Anshit (2024) studied theory and applications of fuzzy sets. In Chang (1985), Zimmermann (1985), Papageorgiou (1985), Lee (2000) and Yan (2010) and also Shi (2007), Kandil (2012) and Akray (2013) studies fuzzy topological nature and its properties, and also in Alkali (2018), Singh (2014), Isah (2019), Isah (2021), Thangathamizh (2024) and Anshit (2024) studies alpha cut levels in fuzzy set and some of its properties were been introduced. The theory of fuzzy set on permutation was first introduced by Aremu (2017), and subsequently by Hassan (2019), Alhassan (2021) where some subsets of Sn were analyzed and many properties were been investigated such as alpha level set, Support (Supp) of fuzzy set, in this paper, the generalization of fuzzy set on permutation Sn was intended to be introduced and alongside some theoretical properties and their proofs will be investigated.

2. Materials and Methods

The concepts of Fuzzy set on permutations, alpha level set, core, support and height which are necessary for this study will be presented in this section, some propositions were been studied and proved in other to be able to come up with the generalizations in terms of fuzzy sets on permutations of Sn.

2.1 Preliminaries

2.1.1 Fuzzy Set: A fuzzy set of a set A is a set with fuzzy boundaries, where each element has a membership degree between 0 and 1. It is defined by a membership function $\mu_{\tilde{A}}(x)$ that maps each element to a value between 0 and 1, indicating the degree of membership.

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}, \quad \text{where} \quad \mu_{\widetilde{A}}(x) : X \to [0, 1]$$

2.1.2 Support of Fuzzy Set: The support of a fuzzy set A is the set of elements x in the universe of discourse X such that the membership degree of x in A is non-zero, i.e.

$$\mu_{\widetilde{A}}(x) > 0.$$

2.1.3 Height of Fuzzy Set: The height of a fuzzy set A is the maximum membership degree achieved by any element in the set, i.e.,

$$max(\mu_{\widetilde{A}}(x)).$$

2.1.4 Core of Fuzzy Set: The core of a fuzzy set A is the set of elements x in the universe of discourse X such that the membership degree of x in A is equal to 1 i.e.

$$\mu_{\widetilde{A}}(x) = max(\mu_{\widetilde{A}}(x)) = 1.$$

2.1.5 Alpha Cut Level of Fuzzy Set: The alpha cut level of a fuzzy set A is the set of elements x in the universe of discourse X such that the membership degree of x in A is greater than or equal to alpha, i.e.,

$$\mu_{\widetilde{A}}(x) \geq \alpha.$$

3. Results

In this section, we define a fuzzy set on permutation and investigate some type1 fuzzy variables and their proofs. **Fuzzy Set**: A fuzzy set of a set A is a set with fuzzy boundaries, where each element has a membership degree between 0 and 1. It is defined by a membership function $\mu_{\widetilde{A}}(x)$ that maps each element to a value between 0 and 1, indicating the degree of membership.

3.1 Fuzzy Set on Permutations

The fuzzy set on permutations can be defined using the function below

$$\widetilde{S_n} = \{(a_i, \mu_{a_i}) : a_i \in S_n\}$$

$$\mu_{a_i} = \frac{\sum_{i=1}^{m} \frac{|\Delta_l^f(a_i)|}{m}}{n}$$

Where:

a_i Is the decomposition of two line permutation

m Is the number of disjoint circles after decomposition of two line permutation.

n Is the subscript of S in S_n.

μ_{a_i} is the membership function of a_i.

$|\Delta_l^f(\boldsymbol{a_i})|$ is the range of circles of $\boldsymbol{a_i}$.

Example of fuzzy set on S_n includes

1. For $\boldsymbol{n=3}$

$$\widetilde{\boldsymbol{S_3}} = \{(a_1, 0), (a_2, 0.3), (a_3, 0.7), (a_4, 0.3), (a_5, 0.7), (a_6, 0.3)\}$$

2. For $\boldsymbol{n=4}$

$$\widetilde{\boldsymbol{S_4}} = \{(a_1, 0), (a_2, 0.3), (a_3, 0.5), (a_4, 0.8), (a_5, 0.3), (a_6, 0.5), (a_7, 0.3),$$
$$(a_8, 0.3), (a_9, 0.5), (a_{\{10\}}, 0.5), (a_{\{11\}}, 0.5), (a_{\{12\}}, 0.3), (a_{\{13\}}, 0.8),$$
$$(a_{\{14\}}, 0.3), (a_{\{15\}}, 0.8), (a_{\{16\}}, 0.5), (a_{\{17\}}, 0.5), (a_{\{18\}}, 0.3),$$
$$(a_{\{19\}}, 0.8), (a_{\{20\}}, 0.5), (a_{\{21\}}, 0.8), (a_{\{22\}}, 0.3), (a_{\{23\}}, 0.5),$$
$$(a_{\{24\}}, 0.3)\}$$

3. For $\boldsymbol{n=5}$

$$\widetilde{\boldsymbol{S_5}} = \{(a_1, 0), (a_2, 0.2), (a_3, 0.2), (a_4, 0.2), (a_5, 0.2), (a_6, 0.2), (a_7, 0.2), (a_8, 0.2),$$
$$(a_9, 0.2), (a_{\{10\}}, 0.2), (a_{\{11\}}, 0.4), (a_{\{12\}}, 0.4), (a_{\{13\}}, 0.4), (a_{\{14\}}, 0.4),$$
$$(a_{\{15\}}, 0.4), (a_{\{16\}}, 0.4), (a_{\{17\}}, 0.4), (a_{\{18\}}, 0.4), (a_{\{19\}}, 0.4), (a_{\{20\}}, 0.4),$$
$$(a_{\{21\}}, 0.4), (a_{\{22\}}, 0.4), (a_{\{23\}}, 0.4), (a_{\{24\}}, 0.6), (a_{\{25\}}, 0.6), (a_{\{26\}}, 0.6),$$
$$(a_{\{27\}}, 0.6), (a_{\{28\}}, 0.6), (a_{\{29\}}, 0.6), (a_{\{30\}}, 0.6), (a_{\{31\}}, 0.6), (a_{\{32\}}, 0.6),$$
$$(a_{\{33\}}, 0.6), (a_{\{34\}}, 0.6), (a_{\{35\}}, 0.6), (a_{\{36\}}, 0.8), (a_{\{37\}}, 0.8), (a_{\{38\}}, 0.8),$$
$$(a_{\{39\}}, 0.8), (a_{\{40\}}, 0.8), (a_{\{41\}}, 0.8), (a_{\{42\}}, 0.8), (a_{\{43\}}, 0.8), (a_{\{44\}}, 0.8),$$
$$(a_{\{45\}}, 0.8), (a_{\{46\}}, 0.8), (a_{\{47\}}, 0.8), (a_{\{48\}}, 0.8), (a_{\{49\}}, 0.2), (a_{\{50\}}, 0.2),$$
$$(a_{\{51\}}, 0.2), (a_{\{52\}}, 0.2), (a_{\{53\}}, 0.2), (a_{\{54\}}, 0.2), (a_{\{55\}}, 0.2), (a_{\{56\}}, 0.2),$$
$$(a_{\{57\}}, 0.2), (a_{\{58\}}, 0.2), (a_{\{59\}}, 0.2), (a_{\{60\}}, 0.2), (a_{\{61\}}, 0.2), (a_{\{62\}}, 0.2),$$
$$(a_{\{63\}}, 0.2), (a_{\{64\}}, 0.2), (a_{\{65\}}, 0.2), (a_{\{66\}}, 0.2), (a_{\{67\}}, 0.2), (a_{\{68\}}, 0.2),$$

$$(a_{\{69\}}, 0.2), (a_{\{70\}}, 0.2), (a_{\{71\}}, 0.2), (a_{\{72\}}, 0.2), (a_{\{73\}}, 0.2), (a_{\{74\}}, 0.2),$$

$$(a_{\{75\}}, 0.2), (a_{\{76\}}, 0.2), (a_{\{77\}}, 0.2), (a_{\{78\}}, 0.2), (a_{\{79\}}, 0.2), (a_{\{80\}}, 0.2),$$

$$(a_{\{81\}}, 0.2), (a_{\{82\}}, 0.2), (a_{\{83\}}, 0.2), (a_{\{84\}}, 0.2), (a_{\{85\}}, 0.2), (a_{\{86\}}, 0.2),$$

$$(a_{\{87\}}, 0.2), (a_{\{88\}}, 0.2), (a_{\{89\}}, 0.2), (a_{\{90\}}, 0.2), (a_{\{91\}}, 0.2), (a_{\{92\}}, 0.2),$$

$$(a_{\{93\}}, 0.2), (a_{\{94\}}, 0.2), (a_{\{95\}}, 0.2), (a_{\{96\}}, 0.2), (a_{\{97\}}, 0.2), (a_{\{98\}}, 0.2),$$

$$(a_{\{99\}}, 0.2), (a_{\{100\}}, 0.2), (a_{\{101\}}, 0.2), (a_{\{102\}}, 0.2), (a_{\{103\}}, 0.2),$$

$$(a_{\{104\}}, 0.2), (a_{\{105\}}, 0.2), (a_{\{106\}}, 0.2), (a_{\{107\}}, 0.2), (a_{\{108\}}, 0.2),$$

$$(a_{\{109\}}, 0.2), (a_{\{110\}}, 0.2), (a_{\{111\}}, 0.2), (a_{\{112\}}, 0.2), (a_{\{113\}}, 0.2),$$

$$(a_{\{114\}}, 0.2), (a_{\{115\}}, 0.2), (a_{\{116\}}, 0.2), (a_{\{117\}}, 0.2), (a_{\{118\}}, 0.2),$$

$$(a_{\{119\}}, 0.2), (a_{\{120\}}, 0.2)\}$$

VISUALISATION OF FUZINESS OF PERMUTATIONS OF S_3

The fuzzy set of permutations of is given below, and graphically illustrated

$$\widetilde{S_3} = \{(a_1, 0), (a_2, 0.3), (a_3, 0.7), (a_4, 0.3), (a_5, 0.7), (a_6, 0.3)\}$$

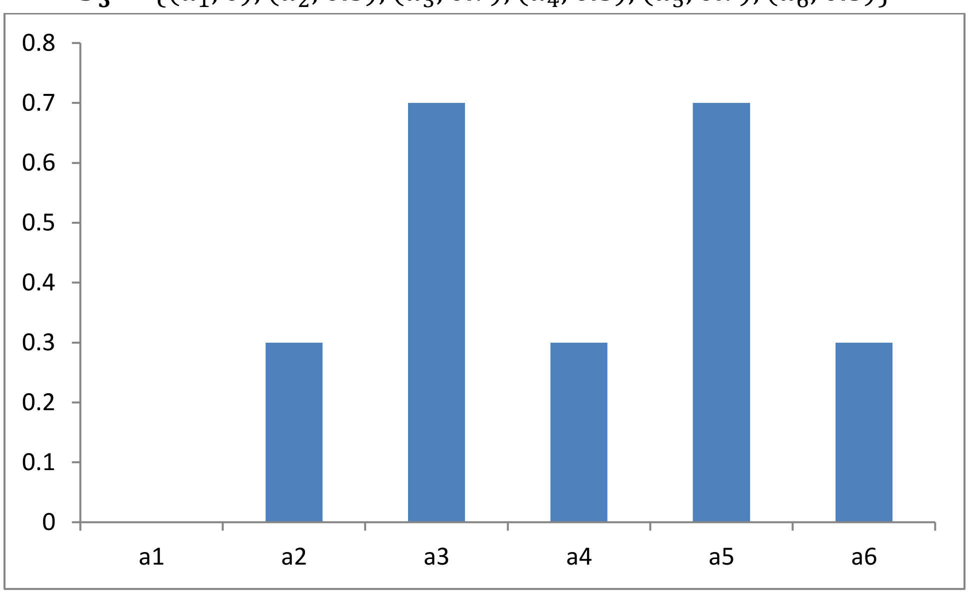

Figure 1. Bar Chat Representation of Fuzzy Permutations of S_3

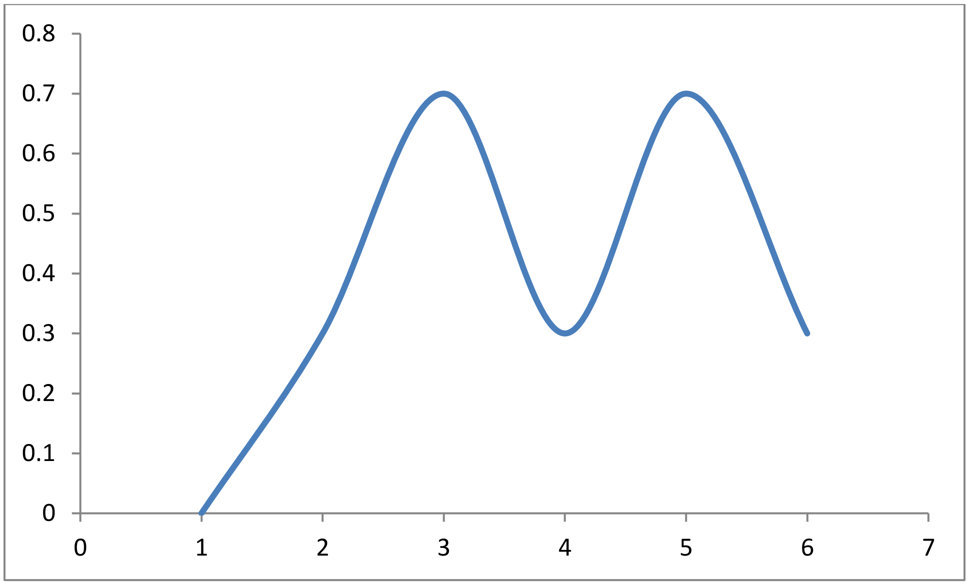

Figure 2. Representation of Fuzzy Permutations of S3 on Line Graph

Proposition 3.1. The support of a fuzzy set $\widetilde{\boldsymbol{S_n}}$ is the entire domain excluding the identity permutation.

Proof.

$\widetilde{S_n} = \{(a_i, \mu_{a_i}) : a_i \in S_n\}$

Where

$a_i = a_1, a_2, a_3, \dots, a_n$

And $a_1 = \boldsymbol{e}$ $(identity\ permutation)$

$\rightarrow (a_1, \mu_{a_1}) = (a_1, 0)$

$\rightarrow \mu_{a_1} = 0, \forall\, a_1 \in S_n$

$\therefore Supp(\widetilde{S_n}) = \{(a_i, \mu_{a_i}) : \mu_{a_i} > 0, a_i \in S_n\}$

And since, $\mu_{a_1} = 0, \forall\, a_1 \in S_n$

Then,

$Supp(\widetilde{S_n}) = \{(a_i, \mu_{a_i}) : a_i \in S_n, i = 2,3,4, \dots, n\}$

Proposition 3.2. The alpha level set and strong alpha level set of fuzzy set of permutations of S_n exist for any given n.

Proof.

a. The alpha level set of S_n is a crisp set given by the set below.

$$\widetilde{S_{n\propto}} = \{ a_i \mid \mu_{a_i} \geq \propto, \quad a_i \in S_n \}$$

For $\boldsymbol{n = 3}$

$S_{3(\propto=0)} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.1)} = \{ a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.2)} = \{ a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.3)} = \{a_1, a_2, a_4, , a_6\}$

$S_{3(\propto=0.4)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.5)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.6)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.7)} = \{a_3,\ a_5 \}$

b. The strong alpha level set of S_n is a crisp set given by the set below.

$$\widetilde{S_{n\propto}} = \{ a_i \mid \mu_{a_i} > \propto, \quad a_i \in S_n \}$$

For $\boldsymbol{n = 3}$

$S_{3(\propto=0)} = \{ a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.1)} = \{ a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.2)} = \{ a_2, a_3, a_4, a_5, a_6\}$

$S_{3(\propto=0.3)} = \{a_3,\ a_5\}$

$S_{3(\propto=0.4)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.5)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.6)} = \{a_3,\ a_5 \}$

$S_{3(\propto=0.7)} = \emptyset$

Proposition 3.3. The height of fuzzy set of S_n (permutation group of n elements) for any n is always < 1.

Proof.

The height of fuzzy set of S_n is given by the set below.

$H(\widetilde{S_n}) = max\{\mu_{a_i}, \ \forall a_i \in S_n\}$

Let $\mu_{a_i^*} > \mu_{a_i}$

$\rightarrow \max\{\mu_{a_i}, \ \forall a_i \in S_n\} = \mu_{a_i^*}$

But, $\forall \mu_{a_i} \in \widetilde{S_n}, \ \max(\mu_{a_i}) < 1$

$\therefore \ \forall \mu_{a_i} \in \widetilde{S_n} \ \exists \mu_{a_i^*} > \mu_{a_i} \ \ni \max(\mu_{a_i}) = \mu_{a_i^*}$

$H(\widetilde{S_n}) = \mu_{a_i^*}$

Proposition 3.4. The core of fuzzy set of permutations of S_n is $\emptyset$, for any $n \in S_n$.

Proof.

$Core(\widetilde{S_n}) = \{a_i \mid \mu_{a_i} = 1\}$

Let $max\{\mu_{a_i}, \ \forall a_i \in S_n\} = \mu_{a_i^*}$

$\mu_{a_i^*} > \mu_{a_i}$

$\rightarrow \max\{\mu_{a_i}, \ \forall a_i \in S_n\} = \mu_{a_i^*}$

But, $\forall \mu_{a_i} \in \widetilde{S_n}, \ \max(\mu_{a_i}) < 1$

$\rightarrow \ Core(\widetilde{S_n}) = \emptyset$

4. Conclusion:

This research investigates the fuzzy type 1 properties of the symmetric group $\boldsymbol{Sn}$, utilizing a generalized membership function to examine key characteristics. Specifically, the study investigated the alpha level set and strong alpha level set, support, height, and core of the fuzzy set generated by $\boldsymbol{Sn}$. By applying the constructed membership function, the research revealed an important insights into the fuzzy structure of $\boldsymbol{Sn}$. The findings provide an insight into understanding of the fuzzy properties that emerge when extending the symmetric group to a fuzzy set. The results are important in development of fuzzy group theory and its applications in various mathematical and computational fields. Also, this work contributes to the advancement of fuzzy mathematics and its capacity to capture and analyze uncertainties in permutation-based problems.

5. Recommendations:

This study investigated fuzzy type 1 properties of the symmetric group Sn, laying the groundwork for future research. To further advance the field, it is recommended to explore fuzzy type 2 properties, which consider both membership and non-membership functions for each element in the fuzzy set. This extension will provide a more comprehensive understanding of the fuzzy structure and its applications. By including both membership and non-membership functions, researchers can capture a wider range of uncertainties, leading to more advance and accurate models in fuzzy group theory and its applications.

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