Taylor Wavelets based Galerkin method for the Numerical solution of boundary value problems

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Abstract

Boundary value problems (BVPs) occur frequently in the fields of engineering and science such as gas dynamics, nuclear physics, atomic structures and chemical reactions. In this paper,I proposed Taylor wavelets based Galerkin method (TWGM) for the Numerical solution of BVPs. Here, I used weight functions as Taylor wavelets that are assumed basis elements which allow us to obtain the numerical solution of the BVPs. Obtained numerical solutions using this method are compared with some existing methods and exact solution. Some BVPs are taken to demonstrate the applicability and validity of the proposed method.

Keywords: *Boundary value problems; Taylor wavelets; Galerkin method*.

1. Introduction

In recent years, studies of boundary value problems in the second order ordinary differential equations have been attracted the attention of many mathematicians and physicists. In most cases, we do not always find the exact solutions for these problems via analytical methods. In this paper, we consider boundary value problems (BVPs) of the type

$$\left.\begin{array}{r} y'' + \alpha y' + \beta y = f(x) \\ \text{With boundary conditions} \quad y(0) = a, \; y(1) = b \end{array}\right\}$$

Thus, the need for the development of numerical approaches to find approximate solutions becomes essential. Recently, some of the numerical methods are used for the numerical solutions of differential equations. For example, Legendre wavelet collocation method [1], Laguerre Wavelet-Galerkin Method [3], Wavelet Galerkin solution [3] etc.

The subject of wavelets has received much attention because of the comprehensive mathematical power and the good application potential of wavelets in many interesting physical problems.

Wavelet functions have generated significant interest from both theoretical and applied research over the last few years.

We can look forward to numerical methods based on wavelet bases to be capable to attain good spatial and spectral resolutions. Representation of a smooth function in terms of a series expansion using orthogonal polynomials is a fundamental idea in approximation theory and forms the basis of spectral methods of solution of differential equations with functional arguments. An approach to study differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. The Galerkin method is considered the most widely used in applied mathematics because of its implementation and simplicity [4-5].

The advantage of wavelet-Galerkin method over finite difference or finite element method has led to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Although the wavelet method provided an efficient alternative technique for solving differential equations especially boundary value problems numerically.

In this paper, I developed Taylor wavelet based Galerkin (TWGM) method for the numerical solution for BVPs. This method is based on expanding the solution by Taylor wavelets with unknown coefficients. The properties of Taylor wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the BVPs is obtained.

The organization of the paper is as follows. Taylor wavelets and function approximation is given section 2. Section 3 deals with Taylor wavelet based Galerkin method for the solution of BVPs. Numerical Experiment is given in section 4. Finally, conclusions of the proposed work are discussed in section 5.

2. Taylor wavelets and Function approximation

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, we have the following family of continuous wavelets [6-7]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right), \; a, b \in R \; \& \; a \neq 0$$

If we restrict the parameters a & b to discrete values as

$$a = a_0^{-n} \;,\; b = m\, b_0\, a_0^{-n} \;;\; a_0 > 1, \quad b_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{n,m}(x) = |a_0|^{\frac{1}{2}} \psi\left(a_0^n x - m b_0\right), \; n, m \in Z$$

Where $\psi_{n,m}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ & $b_0 = 1$, then $\psi_{n,m}(x)$ forms an orthonormal basis.

Taylor wavelets are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k+1}{2}} \tilde{T}_m\left(2^{k+1} x - n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0, & Otherwise \end{cases} \tag{2.1}$$

$$\text{with } \tilde{T}_m(x) = \sqrt{2m+1}\, T_m(x) \tag{2.2}$$

The coefficient $\sqrt{2m+1}$ is for normality, k is any positive integer, $n = 1, 2, 3, ..., 2^{k-1}$ is an argument and T_m $\left(m = 0, 1, 2, 3........M-1\right)$ are the well known Taylor polynomials of oreder m which can be defined by $T_m(x) = x^m$.

Taylor polynomials form a complete basis over the interval $[0, 1)$.

For instance, for $k = 1$ and $M = 3$, we get the Taylor wavelet bases are as follows:

$\psi_{1,0}(x) = 1$, $\psi_{1,1}(x) = \sqrt{3}\, x$, $\psi_{1,2}(x) = \sqrt{5}\, x^2$ and so on.

Function approximation:

Suppose $y(x) \in L^2[0, 1)$ is expanded in terms of Fibonacci wavelets as:

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \tag{2.5}$$

Truncating the above infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi_{n,m}(x) \tag{2.6}$$

3. Method of Solution

Consider the boundary value of the problem is of the form,

$$y'' + \alpha y' + \beta y = f(x) \tag{3.1}$$

With boundary conditions $\quad y(0) = a, \quad y(1) = b \quad$ (3.2)

Where α , β are constants and $f(x)$ be a continuous function.

Write the Eq. (3.1) as $\quad R(x) = y'' + \alpha y' + \beta y - f(x) \quad$ (3.3) where $R(x)$ is the residual of the Eq. (3.1) When $R(x) = 0$ for the exact solution and $y(x)$ will satisfy the boundary conditions.

Consider the trail series solution of the Eq. (3.1), $y(x)$ defined over [0, 1) can be expanded as a modified Fibonacci wavelets, satisfying the given boundary conditions which is involving unknown parameter as follows,

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi_{n,m}(x) \tag{3.4}$$

where $c_{n,m}$'s are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher degree Taylor wavelet polynomials.

Differentiating Eq. (3.4) twice with respect to x and substitute the values of in y, y', y'' Eq. (4.3). To find $c_{n,m}$'s we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [8].

i.e. $$\int_0^1 \psi_{1,m}(x)\, R(x)\, dx = 0, \; m = 0, 1, 2, \ldots\ldots$$

then we obtain a system of linear algebraic equations, on solving this system, we get unknown parameters. Then substitute these unknowns in the trail solution i.e. Eq. (3.4), we obtained numerical solution of Eq. (3.1).

In order to know the accuracy of TWGM for the test problems, we use the error measure i.e. maximum absolute error. The maximum absolute error will be calculated by

$$E_{\max} = \max \left| y(x)_e - y(x)_a \right|,$$

where $y(x)_e$ and $y(x)_a$ are exact and approximate solutions respectively.

4. Numerical Experiment

Problem 4.1 First, consider the boundary value problem,

$$y'' - y = x - 1, \quad 0 \leq x \leq 1 \tag{4.1}$$

With boundary conditions: $y(0) = 0, \; y(1) = 0$ (4.2)

The implementation of the Eq. (4.1) as per the method explained in section 3 is as follows:

and its residual can be written as:

$$R(x) = y'' - y - (x - 1) \tag{4.3}$$

Now, choosing the weight function $w(x) = x(1 - x)$ for Tayolr wavelet bases to satisfy the given boundary conditions Eq. (4.2), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1 - x) = x(1 - x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \left(\sqrt{3}\, x\right) x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \left(\sqrt{5}\, x^2\right) x(1-x)$$

Assuming the trail solution of Eq. (4.1) for $k = 1$ and $m = 2$ is given by

$$y(x) = c_{1,0}\, \psi_{1,0}(x) + c_{1,1}\, \psi_{1,1}(x) + c_{1,2}\, \psi_{1,2}(x) \tag{4.4}$$

Then the Eq. (4.4) becomes

$$y(x) = c_{1,0}\left\{x(1-x)\right\} + c_{1,1}\left\{\left(\sqrt{3}\, x\right)x(1-x)\right\} + c_{1,2}\left\{\left(\sqrt{5}\, x^2\right)x(1-x)\right\}$$

$$\Rightarrow y(x) = c_{1,0}\left(x - x^2\right) + c_{1,1}\left\{\sqrt{3}\left(x^2 - x^3\right)\right\} + c_{1,2}\left\{\sqrt{5}\left(x^3 - x^4\right)\right\} \tag{4.5}$$

Differentiating Eq. (4.5) twice w.r.t. x and substitute the values of y , y'' in Eq. (4.3), we get the residual of Eq. (4.1). The "weight functions" are the same as the basis functions.

Then by the weighted Galerkin method, we consider the following:

$$\int_0^1 \psi_{1,j}(x)\, R(x)\, dx = 0, \quad j = 0, 1, 2 \tag{4.6}$$

For $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\, R(x)\, dx &= 0 \\ \int_0^1 \psi_{1,1}(x)\, R(x)\, dx &= 0 \\ \int_0^1 \psi_{1,2}(x)\, R(x)\, dx &= 0 \end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), we have system of algebraic equations with unknown coefficients i.e. $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. Solving this by Gauss elimination method, we obtain the values for $c_{1,0} = 0.3128$, $c_{1,1} = -0.1065$ and $c_{1,2} = 0.0098$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 1 and numerical solution with exact solution of Eq. (4.1) is $y(x) = -\left(\frac{1}{1 - e^2}\right)e^x + \left(\frac{e^2}{1 - e^2}\right)e^{-x} - x + 1$ [9] in figure 1.

Table 1. Comparison of numerical solution and absolute error with exact solution of the problem 4.1.

x	Numerical solution			Exact solution	Absolute error		
	Ref **[3]**	Ref **[9]**	TWGM		Ref **[3]**	Ref **[9]**	TWGM

0.1	0.0276352	0.0264712	0.0265116	0.0265183	1.12e-03	4.70e-05	6.70e-06
0.2	0.0453501	0.0443444	0.0442854	0.0442945	1.06e-03	5.00e-05	9.10e-06
0.3	0.0545619	0.0546184	0.0544810	0.0545074	5.50e-05	1.10e-04	2.60e-05
0.4	0.0566876	0.0583436	0.0582050	0.0582599	1.57e-03	8.40e-05	5.50e-05
0.5	0.0531447	0.0565875	0.0565917	0.0565906	3.45e-03	3.10e-06	1.10e-06
0.6	0.0453501	0.0504023	0.0504026	0.0504834	5.13e-03	8.10e-05	8.10e-05
0.7	0.0347212	0.0407912	0.0408268	0.0408782	6.16e-03	8.70e-05	5.10e-05
0.8	0.0226751	0.0286751	0.0286806	0.0286795	6.00e-03	4.40e-06	1.10e-06
0.9	0.0106289	0.0148592	0.0148080	0.0147663	4.14e-03	9.30e-05	4.20e-05

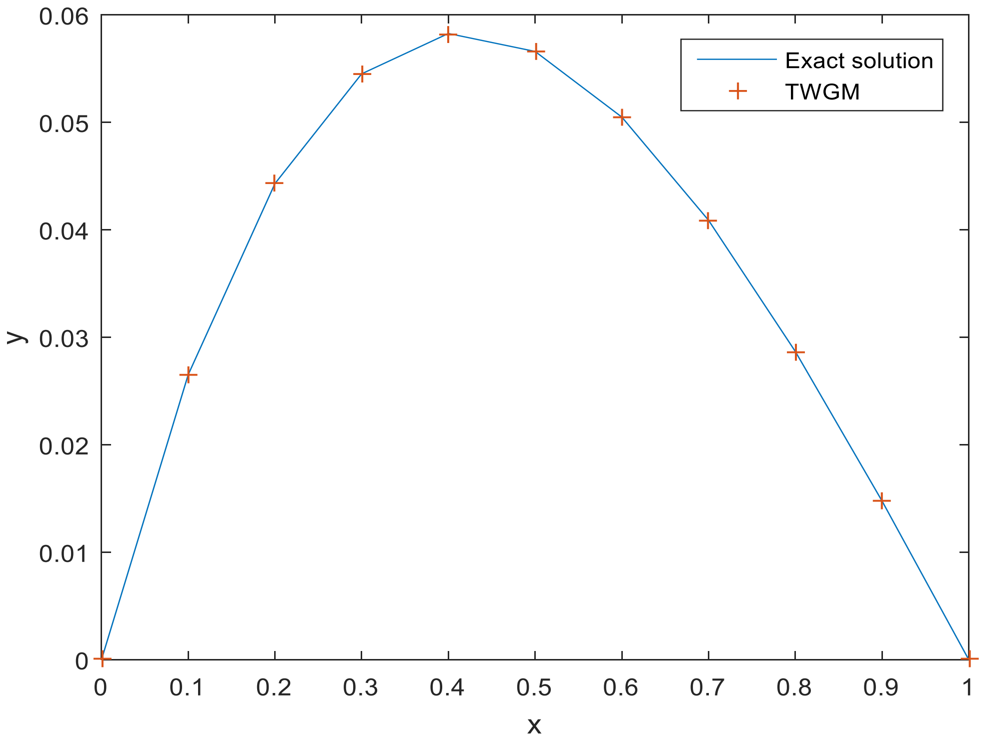

Fig. 1. Comparison of numerical solution with exact solution of the problem 4.1.

Problem 4.2 Next, consider the boundary value problem,

$$\frac{d^2 y}{d x^2} - \frac{d y}{d x} = -\left(e^{x-1} + 1\right), \quad 0 \leq x \leq 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.9)

As explained in section 3 and in the previous problem, we obtain the values of $c_{1,0} = 0.5265$, $c_{1,1} = 0.1495$ and $c_{1,2} = 0.1437$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 2 and numerical solution with exact solution of Eq. (4.1) is $y(x) = x\left(1 - e^{x-1}\right)$ [10] in figure 2.

Table 2. Comparison of numerical solution and absolute error with exact solution

of the problem 4.2.

x	Numerical solution			Exact solution	Absolute error		
	FDM	**Ref[10]**	TWGM		FDM	**Ref[10]**	TWGM
0.1	0.061948	0.0593827	0.059366	0.059343	2.61e-03	3.97e-05	2.50e-05
0.2	0.115151	0.1102340	0.110174	0.110134	5.02e-03	1.00e-04	4.00e-05
0.3	0.158162	0.1512000	0.150963	0.151024	7.14e-03	1.76e-04	6.10e-05
0.4	0.189323	0.1806167	0.180380	0.180475	8.85e-03	1.42e-04	9.50e-05
0.5	0.206737	0.1969833	0.196681	0.196735	1.00e-02	2.48e-04	5.40e-05
0.6	0.208235	0.1980833	0.197831	0.197808	1.04e-02	2.75e-04	2.30e-05
0.7	0.191342	0.1816552	0.181502	0.181427	9.92e-03	2.28e-04	7.50e-05
0.8	0.153228	0.1452000	0.145075	0.145015	8.21e-03	1.85e-04	6.00e-05
0.9	0.090672	0.0857100	0.085642	0.085646	5.03e-03	6.40e-05	4.00e-06

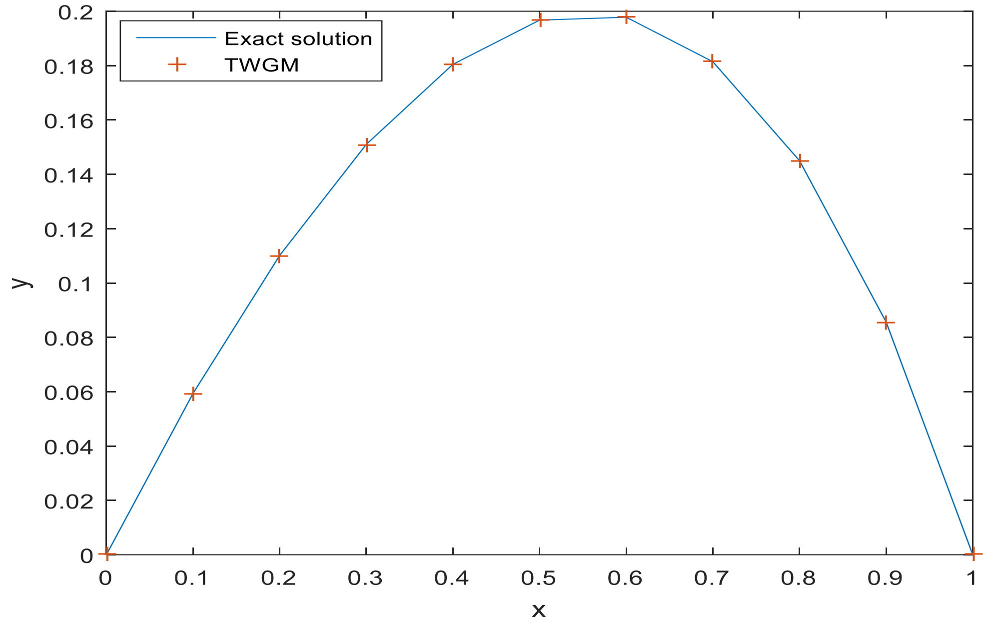

Fig. 2. Comparison of numerical solution with exact solution of the problem 4.2.

5. Conclusions

In this paper, I proposed the Taylor wavelet based Galerkin method for the numerical solution of boundary value problems. Form the above tables and figures, we observed that:

a) The numerical solutions obtained by the proposed method are better than the finite difference method (FDM) & some existing methods and nearer to the exact solution.

b) The absolute error produced by this method is significantly lower when compared to FDM & some existing methods.

Hence the Taylor wavelet based Galerkin method is very effective for solving boundary value problems.

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