Global existence and decay of solutions for a variable coefficients Petrovsky equation with delay term

[1]Ayşe Fidan & [2,*]Erhan Pişkin

[1]*Department of Mathematics, Institute of Natural and Applied Sciences*
Dicle University, Diyarbakır, Turkey

[2]*Department of Mathematics*
Dicle University, Diyarbakır, Turkey
[*]*Corresponding Author Email: episkin@dicle.edu.tr*

Abstract

This work focuses on the Petrovsky equation with a variable coefficient delay term. Firstly by using the Faedo-Galerkin method, we prove the global existence of the solution under suitable conditions. Later, we establish a decay estimate for the energy by using the Martinez lemma.

Keywords: *Decay, Global existence, Petrovsky equation, Variable coefficients, time-varying delay*
Mathematics Subject Classification (2020): 35B05, 35A0, 35L35, 35N10, 35B40

1. INTRODUCTION

1.1 Model

We consider Petrovsky equation involving with initial and boundary conditions.

$$\begin{cases} u_{tt} + \Delta^2 u + \beta_1(t) u_t(x,t) + \beta_2(t) u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = \Delta u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t > 0, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ $(n \geq 1)$ a domain that is limited by a smooth boundary $\partial\Omega$. Also, $\tau > 0$ is the time-varying delay and u_0, u_1 and f_0 are initial data. β_1 and β_2 are positive functions.

Time delays frequently arise in various real-world issues, such as economic phenomena, biological, chemical, thermal, and physical processes. Additionally, it is understood that even a very small delay can lead to the destabilization of a system that would otherwise be uniformly asymptotically stable without the delay, unless there are control measures or additional conditions in place. As such, delays serve as a potential source of instability Fridman [6] and Kafini [11].

1.2 Literature review

Benaissa et al. in [3] examined time-varying delay equation of the form

$$u_{tt}-\Delta u+\mu_1(t)u_t(x,t)+\mu_2(t)u_t(x,t-\tau)=0$$

and the existence of solutions has been proven, and the decay rate estimates for the energy were obtained.

Meyvacı [13] discussed the subsequent fourth-order wave equation

$$u_{tt}+\Delta^2 u+u+\sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t)+\sigma_2(t)|u_t(x,t-\tau)|^{2m-2}u_t(x,t-\tau)=0.$$

She established the decay estimate.

Guo and Li [8] considered the Petrovsky term evolution equation solutions with strong damping

$$u_{tt}+\Delta^2 u-\Delta u-\Delta u_t+\alpha(t)u_t=|u|^{p-2}u.$$

They utilized an energy estimation technique to derive the minimum possible blow-up time.

Wu [18] considered with variable coefficients

$$u_{tt}+\Delta^2 u-\Delta u-\omega\Delta u_t+\alpha(t)u_t=|u|^{p-2}u$$

and obtained the blow-up result with lower and upper bounded.

Zheng et al. [19] considered coefficients

$$u_{tt}+\Delta^2 u+k_1(t)|u_t|^{m-2}u_t=k_2(t)|u|^{p-2}u$$

and has investigated existence of a solution to the Petrovsky type equation based on time-varying delay.

Problems with variable coefficients have been handled carefully in several papers, some results relating the existence, blow up and stability have been found [2, 4, 5, 9, 16, 17].

1.3 Main contribution

We will show the global existence theorem by using the Faedo-Galerkin's method. Also, we establish the decay result of solutions by using the Martinez lemma.

The rest of the work is as follows: In Section 2, we give some assumptions needed in this work. In Section 3, we prove the global existence theorem by the Faedo-Galerkin method. The decay result is proved in Section 4.

2 PRELIMINARIES

Throughout the paper, we give some materials, lemmas and theorems.

We make the following assumptions on $\beta_1(t)$ and $\beta_2(t)$:

(H1) $\beta_1 : R^+ \to (0,+\infty)$ is a nonincreasing function on $C^1(R^+)$ convince

$$\left|\frac{\beta_1'(t)}{\beta_1(t)}\right| \le M, \tag{2}$$

(**H2**) $\beta_2 : R^+ \to R$ is a function on $C^1(R^+)$, which mayn't be positive or monotone,

$$\begin{aligned} |\beta_2(t)| &\le \mu\, \beta_1(t), \\ |\beta_2'(t)| &\le \eta\, \beta_1(t), \end{aligned} \tag{3}$$

$$0 < \mu < 1 \quad \text{and} \quad \eta > 0. \tag{4}$$

Now, we give the lemma which will be used in the proof of the decay sense.

Lemma 1 [12]. Let $E:\ R^+ \to R^+$ be a nonincreasing function and $\phi:\ R^+ \to R^+$ a strictly increasing C^1 function such that the

$$\phi(0) = 0 \ \text{and}\ \phi(t) \to \infty \ \text{as}\ t \to +\infty.$$

Let exists $\sigma > -1$ and $\omega > 0$ so that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt < \frac{1}{\omega} E^{\sigma}(0)E(S),\ \ 0 \le S < +\infty. \tag{5}$$

Later

$$E(t) = 0, \quad \forall t \ge \frac{E^{\sigma}(0)}{\omega|\sigma|},\ \ \text{if}\ \ -1 < \sigma < 0, \tag{6}$$

$$E(t) \le E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}}, \quad \forall t \ge 0,\ \text{if}\ \ \sigma > 0, \tag{7}$$

$$E(t) \le E(0)e^{1-\omega\phi(t)}, \quad \forall t \ge 0,\ \text{if}\ \ \sigma = 0. \tag{8}$$

For studying problem (10), we present a new function z as follows ([14]):

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \Omega, \rho \in (0,1), t > 0. \tag{9}$$

Then, we get

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ \text{in}\ \Omega \times (0,1) \times (0,+\infty). \tag{10}$$

Therefore, Problem (1) can be transformed into

$$\begin{cases} u_{tt} + \Delta^2 u + \beta_1(t)u_t(x,t) + \beta_2(t)z(x,1,t) = 0, & x \in \Omega,\ t > 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Omega,\ \rho \in (0,1), t > 0, \\ u(x,t) = \Delta u = 0, & x \in \partial\Omega,\ t > 0, \\ z(x,0,t) = u_t(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,\rho,0) = f_0(x,-\tau\rho), & x \in \Omega,\ \rho \in (0,1). \end{cases} \tag{11}$$

Let $\bar{\xi}$ be a fixed positive value such that

$$\tau\mu < \bar{\xi} < \tau(2-\mu). \tag{12}$$

The functional E of problem (21) is as follows:

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx, \tag{13}$$

here $\xi(t) = \bar{\xi}\beta_1(t)$.

Lemma 2 Assume that (u,z) is a regular solution to the Problem (11). Then the energy of Problem (11) defined by (13) satisfies

$$\begin{aligned} E'(t) &\le -\left(\beta_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\beta_2(t)|}{2}\right)\|u_t\|^2 - \left(\frac{\xi(t)}{2\tau} - \frac{|\beta_2(t)|}{2}\right)\|z(x,1,t)\|^2 \\ &\le 0. \end{aligned} \tag{14}$$

Proof Multiply the first equation of (11) by u_t integrate it over $\Omega,$ apply Green's formula, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_t\|^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\Delta u\|^2 + \beta_1(t)\|u_t\|^2 + \beta_2(t)\int_\Omega u_t(x,t-\tau)u_t(x,t)\,dx = 0. \tag{15}$$

Multiplying the second equation of (11) by $\zeta(t)z$ and integrate it over $\Omega\times(0,1)$, we have

$$\xi(t)\tau\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho dx + \xi(t)\int_\Omega\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)\,d\rho dx = 0. \tag{16}$$

This results in

$$\frac{\xi(t)\tau}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx + \frac{\xi(t)}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,t)\,d\rho dx = 0,$$

which yields

$$\begin{aligned} &\frac{\tau}{2}\left[\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx\right) - \xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx\right] \\ &+ \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx\right) \\ &= \frac{\tau}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx + \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx. \end{aligned} \tag{17}$$

Combination of (15) and (17) leads to

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right]\\&=-\beta_1(t)\|u_t(x,t)\|^2-\beta_2(t)\int_\Omega z(x,1,t)u_t(x,t)dx\\&+\frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi(t)}{2\tau}\|u_t(x,t)\|^2.\end{aligned}$$

Recalling the definition of $E(t)$ in (13), we can obtain

$$\begin{aligned}E'(t)&=-\left(\beta_1(t)-\frac{\xi(t)}{2\tau}\right)\|u_t\|^2-\beta_2(t)\int_\Omega z(x,1,t)u_t(x,t)dx\\&\quad+\frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx.\\&\le-\left(\beta_1(t)-\frac{\xi(t)}{2\tau}\right)\|u_t\|^2-\beta_2(t)\int_\Omega z(x,1,t)u_t(x,t)dx\\&\quad-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx.\end{aligned}\tag{18}$$

Using the Young inequality, we can obtain

$$\int_\Omega z(x,1,t)u_t(x,t)dx\le\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|z(x,1,t)\|^2.\tag{19}$$

Substituting (33) into (32), we get

$$\begin{aligned}E'(t)&\le-\left(\beta_1(t)-\frac{\xi(t)}{2\tau}-\frac{|\beta_2(t)|}{2}\right)\|u_t\|^2-\left(\frac{\xi(t)}{2\tau}-\frac{|\beta_2(t)|}{2}\right)\|z(x,1,t)\|^2\\&\le-\beta_1(t)\left(1-\frac{\bar{\xi}}{2\tau}-\frac{\beta}{2}\right)\|u_t\|^2-\beta_1(t)\left(\frac{\bar{\xi}}{2\tau}-\frac{\beta}{2}\right)\|z(x,1,t)\|^2\le 0.\end{aligned}\tag{20}$$

The proof of the lemma is now finished.

2 GLOBAL EXISTENCE

In this part, we shall establish global existence of the solutions for problem (1).

Theorem 3 Suppose that $u_0\in H^2(\Omega)\cap H_0^2(\Omega)$, $u_1\in H^2(\Omega)$, $f_0\in L^2(\Omega;H^2(0,1))$ and

$$f_0(.,0)=u_1$$

hold. Also, suppose that (H1)-(H2) hold. Then, there exists $T>0$ so that problem (10) has unique global weak solution u in the class

$$u\in L^\infty_{loc}((-\tau,\infty);H^2(\Omega)\cap H_0^2(\Omega)),u_t\in L^\infty_{loc}((-\tau,\infty);H_0^2(\Omega)),u_{tt}\in L^\infty_{loc}((-\tau,\infty);L^2(\Omega)).$$

Proof We will prove the global existence theorem using the Faedo-Galerkin method. Let $T>0$ be a fixed parameter and let V_k be the space generated by the set $\{w_1,w_2,...,w_k\}$, where the set

$\{w_k, k \in N\}$ forms a Hilbertian basis in $H^2(\Omega) \cap H^2(\Omega)$.

Currently, we define, for $1 \le j \le k$, the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

We can expand $\phi_j(x,0)$ using $\phi_j(x,\rho)$ over $L^2(\Omega \times (0,1))$, so that the set $(\phi_j)_j$ forms a basis for $L^2(\Omega; H^2(0,1))$, denoted as Z_k, which is the space generated by $\{\phi_1, \phi_2, ..., \phi_k\}$, where k is a natural number.

We formulate estimate solutions by (u_k, z_k), $k = 1,2,3,...$, gets

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t) w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk}(t) \phi_j,$$

Here $g_{jk}(t)$ and h_{jk} $(j = \overline{1,k})$ are decided by the using system of ODE:

$$\begin{cases} (u_{ktt}, w_j) + (\Delta u_k, \Delta w_j) + \beta_1(t)(u_{kt}, w_j) + \beta_2(t)(z_t(.,1), w_j) = 0, \\ 1 \le j \le k, \\ z_k(x,0,t) = u_{kt}(x,t), \end{cases} \tag{21}$$

related to initial conditionals

$$u_k(t) = u_{0k}(t) = \sum_{j=1}^{k} (u_0, w_j) w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^2(\Omega) \text{ as } k \to +\infty, \tag{22}$$

$$u_{kt}(0) = u_{1k}(t) = \sum_{j=1}^{k} (u_1, w_j) w_j \to u_1 \text{ in } H_0^2(\Omega) \text{ as } k \to +\infty, \tag{23}$$

$$\begin{cases} (\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \\ \quad 1 \le j \le k, \end{cases} \tag{24}$$

$$z_k(\rho,0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } L^2(\Omega; H_0^2(0,1)) \text{ as } k \to +\infty. \tag{25}$$

Through the ODE's theory, the system (21) -(25) possesses a unique solution, which is extended to a maximum time interval $[0, T_k)$ (where $0 < T_k \le +\infty$) through the utilization of Zorn lemma. It should be noted that $u_k(t)$ belongs to the C^2 class.

Now, we get a prior obtain for the solution of the system (21) -(25). Later, the solutions interval $[0, T_k)$ for $t > 0$ can be extended.

A prior estimate: Once the sequences u_{0k}, u_{1k} and z_{0k} converge, we can determine a positive constant C, which remains independent of k, by utilizing equation (14).

$$E_k(t)+\int_0^t a_1(s)\|u_{kt}(s)\|_2^2\,ds+\int_0^t a_2(s)\|z_k(x,1,s)\|^2\,ds\le E_k(0)\le C \tag{26}$$

and

$$E_k(t)=\frac{1}{2}\|u_{kt}(t)\|^2+\frac{1}{2}\|\Delta u_k(t)\|^2+\frac{\zeta(t)}{2}\int_\Omega\int_0^1 z_k^2(x,\rho,t)\,d\rho dx, \tag{27}$$

$$a_1(t)=\beta_1(t)\left(1-\frac{\bar{\xi}}{2\tau}-\frac{\beta}{2}\right)\ \ and\ \ a_2(t)=\beta_1(t)\left(\frac{\bar{\xi}}{2\tau}-\frac{\beta}{2}\right). \tag{28}$$

This implies that the solution (u_k, z_k) exists globally within the interval $[0,+\infty)$.
Approximate (26) the result of

$$\begin{aligned}
&(u_k)\ \text{ is bounded in } L^\infty_{loc}\left(0,\infty;H^2(\Omega)\right),\\
&(u_{kt})\ \text{ is bounded in } L^\infty_{loc}\left(0,\infty;L^2(\Omega)\right),\\
&\beta_1(t)\left(z_k^2(x,\rho,t)\right) \text{ is bounded in } L^1(\Omega\times(0,T)),\\
&\beta_1(t)z_k(x,0,t) \text{ is bounded in } L^\infty_{loc}\left(0,\infty;L^1(\Omega\times(0,1))\right),\\
&\beta_1(t)z_k(x,1,t) \text{ is bounded in } L^\infty_{loc}\left(0,\infty;L^1(\Omega\times(0,1))\right).
\end{aligned} \tag{29}$$

The second estimate: We first approximate $u_{ktt}(0)$. Replacement w_j by $u_{ktt}(0)$ in (21) and taking $t=0$, we have:

$$\begin{aligned}
\|u_{ktt}(0)\|_2 &\le \|\Delta u_{0k}\|+\beta_1(0)\|u_{1k}\|+|\beta_2(0)|\,\|z_{0k}\|\\
&\le \|\Delta u_0\|+\beta_1(0)\|u_1\|+|\beta_2(0)|\,\|z_0\|\\
&\le C.
\end{aligned} \tag{30}$$

Differentiating (21) in terms of t, we obtain

$$(u_{kttt}(t)+\Delta u_{kt}(t)+\beta_1(t)u_{ktt}(t)+\beta_1'(t)u_{kt}(t)+\beta_2(t)z_{kt}(1,t)+\beta_2'(t)z_k(1,t),w_j)=0. \tag{31}$$

Multiplying by $g_{jktt}(t)$, summing over j from 1 to k, we can deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_{ktt}(t)\|^2+\|\Delta u_{kt}(t)\|^2\right)+\beta_1(t)\int_\Omega u_{ktt}^2(t)dx+\beta_1'(t)\int_\Omega u_{ktt}(t)u_{kt}(t)dx\\
&+\beta_2(t)\int_\Omega u_{ktt}(t)z_{kt}(x,1,t)+\beta_2'(t)\int_\Omega u_{ktt}(t)z_k(x,1,t)dx=0.
\end{aligned} \tag{32}$$

Differentiating equation (24) with respect to t

$$\left(\tau z_{ktt}(t)+\frac{\partial}{\partial\rho}z_{k\rho},\phi_j\right)=0.$$

By multiplying the above equation above with $h_{jkt}(t)$, summing-over j from 1 to k, as a result

$$\frac{\tau}{2}\frac{d}{dt}\|z_{kt}(t)\|^2+\frac{1}{2}\frac{d}{d\rho}\|z_{kt}\|^2=0. \tag{33}$$

Combining (32) and (33), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_{ktt}(t)\|^2+\|\Delta u_{kt}(t)\|^2+\tau\int_0^1\|z_{kt}(x,\rho,t)\|^2_{L^2(\Omega)}d\rho\right)\\
&+\beta_1(t)\int_\Omega u_{ktt}^2(t)dx+\frac{1}{2}\int_\Omega|z_{kt}(x,1,t)|^2dx\\
&=-\beta_2(t)\int_\Omega u_{ktt}(t)z_{kt}(x,1,t)dx-\beta_1'(t)\int_\Omega u_{ktt}(t)u_{kt}(t)dx\\
&-\beta_2'(t)\int_\Omega u_{ktt}(t)z_k(x,1,t)dx+\frac{1}{2}\|u_{ktt}(t)\|^2
\end{aligned} \tag{34}$$

By employing $(H1)$ and $(H2)$, along with the Cauchy-Schwarz's inequality and Young inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_{ktt}(t)\|^2+\|\Delta u_{kt}(t)\|^2+\tau\int_0^1\|z_{kt}(x,\rho,t)\|^2_{L^2(\Omega)}d\rho\right)+\beta_1(t)\int_\Omega u_{ktt}^2(t)dx+\frac{1}{2}\int_\Omega|z_{kt}(x,1,t)|^2dx\\
&\le|\beta_2(t)|\|u_{ktt}(t)\|\|z_{kt}(x,1,t)\|+|\beta_1'(t)|\|u_{ktt}(t)\|\|u_{kt}(t)\|+\beta_2'(t)\|u_{ktt}(t)\|\|z_k(x,1,t)\|+\frac{1}{2}\|u_{ktt}(t)\|^2\\
&\le\frac{|\beta_2(t)|^2}{2}\|u_{ktt}(t)\|^2+\frac{1}{2}\|z_{kt}(x,1,t)\|^2+\frac{|\beta_1'(t)|}{4}\|u_{ktt}(t)\|^2+|\beta_1'(t)|\|u_{kt}(t)\|^2+\frac{|\beta_2'(t)|}{4}\|u_{ktt}(t)\|^2+|\beta_2'(t)|\|z_k(x,1,t)\|^2+\frac{1}{2}\|u_{ktt}(t)\|^2\\
&\le c'\|u_{ktt}(t)\|^2+|\beta_1'(t)|\|u_{kt}(t)\|^2+|\beta_2'(t)|\|z_k(x,1,t)\|^2+\frac{1}{2}\|z_{kt}(x,1,t)\|^2\\
&\le c'\|u_{ktt}(t)\|^2+M\beta_1(t)\|u_{kt}(t)\|^2+\overline{M}\beta_1(t)\|z_k(x,1,t)\|^2+\frac{1}{2}\|z_{kt}(x,1,t)\|^2.
\end{aligned}$$

After integrating the last inequality over the interval $(0,t)$ and utilizing the equation (26), we get

$$\begin{aligned}
&\left(\|u_{ktt}(t)\|^2+\|\Delta u_{kt}(t)\|^2+\tau\int_0^1\|z_{kt}(x,\rho,t)\|^2_{L^2(\Omega)}d\rho\right)\\
&\le\left(\|u_{ktt}(0)\|^2+\|\Delta u_{kt}(0)\|^2+\tau\int_0^1\|z_{kt}(x,\rho,0)\|^2_{L^2(\Omega)}d\rho\right)+2M\int_0^t\beta_1(s)\|u_{kt}(s)\|^2ds\\
&+2\overline{M}\int_0^t\beta_1(s)\|z_k(x,1,s)\|^2ds+2c'\int_0^t\|u_{ktt}(s)\|^2ds\\
&\le C+C'\int_0^t\left(\|u_{ktt}(s)\|^2+\|\Delta u_{kt}(s)\|^2+\tau\int_0^1\|z_{kt}(x,\rho,s)\|^2_{L^2(\Omega)}d\rho\right)ds.
\end{aligned}$$

Then, Applying Gronwall's lemma, we conclude

$$\|u_{ktt}(t)\|_2^2 + \|\Delta u_{kt}(t)\|_2^2 + \tau \int_0^1 \|z_{kt}(x,\rho,t)\|_{L^2(\Omega)}^2 d\rho \le Ce^{C'T} \tag{35}$$

for all $t \in R^+$, from here we deduce that

$$\begin{aligned} &(u_{ktt}) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega)),\\ &(u_{kt}) \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^1(\Omega)),\\ &\tau z_{kt} \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega\times(0,1))). \end{aligned} \tag{36}$$

Appliying Dunford Pettis theorem, we deduce using to (29) and (36), we can substitute the sequence u_k with a subsequence if needed, in such a way that

$$\begin{aligned} &u_k \to u \text{ weak star in } L^\infty_{loc}(0,\infty; H^2(\Omega)\cap H^2(\Omega)),\\ &u_{kt} \to u_t \text{ weak star in } L^\infty_{loc}(0,\infty; H^2(\Omega)),\\ &u_{ktt} \to u_{tt} \text{ weak star in } L^\infty_{loc}(0,\infty; L^2(\Omega)),\\ &u_{kt} \to \chi \text{ weak in } L^2(\Omega\times(0,1); \beta_1(t)), \end{aligned} \tag{37}$$

$$\begin{aligned} &z_k \to z \text{ weak star in } L^\infty_{loc}(0,\infty; H^2(\Omega)\cap L^2(0,1)),\\ &z_{kt} \to z_t \text{ weak star in } L^\infty_{loc}(0,\infty; H^2(\Omega)\cap L^2(0,1)),\\ &z_k(x,1,t) \to \psi \text{ weak in } L^2(\Omega\times(0,T), \beta_1(t)). \end{aligned} \tag{38}$$

Then

$$\begin{aligned} &u \in L^\infty(0,T; H^2(\Omega)\cap H^2(\Omega)), z \in L^\infty(0,T; L^2(\Omega\times(0,1))),\\ &\chi \in L^2(\Omega\times(0,T); \beta_1(t)), \psi \in L^2(\Omega\times(0,T), \beta_1(t)), \end{aligned} \tag{39}$$

for every $T \ge 0$. We must prove that u satisfies equation (1).

Using to (35) we established that u_{kt} is limited in $L^\infty(0,T; H^2(\Omega))$. Then (u_{kt}) is limited in $L^2(0,T; H^2(\Omega))$. Since (u_{ktt}) is limited in $L^\infty(0,T; L^2(\Omega))$, then it is bounded within $L^2(0,T; L^2(\Omega))$. As a result, (u_{kt}) is limited in $H^1(Q)$.

So $H^1(Q) \to L^2(Q)$ is compact the embedding, from Aubin-Lions theorem [10], we can extract a subsequence (u_ς) to (u_k) so that

$$u_{\varsigma t} \to u_t \text{ strongly in } L^2(Q). \tag{40}$$

Thus

$$u_{\varsigma t} \to u_t \text{ strongly and a.e. in } Q. \tag{41}$$

Likewise we prove

$$z_\varsigma \to z \text{ strongly in } L^2(\Omega\times(0,1)\times(0,T)), \tag{42}$$

$$z_{\varsigma} \to z \text{ strongly and a.e. in } \Omega \times (0,1) \times (0,T) \tag{43}$$

It immediately follows from (53), (54), (55), (56) and (57) for every constant $v \in L^2(0,T;L^2(\Omega))$ and $\omega \in L^2(0,T;L^2(\Omega)\times(0,1))$

$$\int_0^t \int_\Omega \left(u_{tt\varsigma} + \Delta^2 u_{\varsigma} + \beta_1(t) u_{t\varsigma} + \beta_2(t) z_{\varsigma}\right) v dx dt$$
$$\to \int_0^t \int_\Omega \left(u_{tt} + \Delta^2 u + \beta_1(t) u_t + \beta_2(t) z\right) v dx dt,$$

$$\int_0^T \int_0^1 \int_\Omega \left(\tau z_{t\varsigma} + \frac{\partial}{\partial \rho} z_{\varsigma}\right) \omega dx d\rho dt \to \int_0^T \int_0^1 \int_\Omega \left(\tau z_t + \frac{\partial}{\partial \rho} z\right) \omega dx d\rho dt$$

as $\varsigma \to +\infty$. Therefore the problem (10) admits of a global weak solution u.

Uniqueness: Let (u_1, z_1) et (u_2, z_2) are two solutions of problem (21) and $(\omega, \varpi) = (u_1, z_1) - (u_2, z_2)$ satisfies

$$\begin{cases} \omega_{tt}(x,t) + \Delta^2 \omega(x,t) + \beta_1(t)\omega_t(x,t) + \beta_2(t)\varpi(x,1,t) = 0 & in\, \Omega \times (0,+\infty), \\ \tau \varpi_t(x,\rho,t) + \varpi_\rho(x,\rho,t) = 0, & in\, \Omega \times (0,+\infty) \times (0,+\infty), \\ \omega(x,t) = 0, \quad \Delta u = 0 & on\ \partial\Omega \times (0,+\infty), \\ \varpi(x,0,t) = \omega_t(x,t), & on\ \partial\Omega \times (0,+\infty), \\ \omega(x,0) = 0, \ \ \omega_t(x,0) = 0, & in\, \Omega, \\ \varpi(x,\rho,0) = 0, & on\ \Omega \times (0,1). \end{cases}$$

We multiply the first equation in (62) by ω_t, and using integration by part over Ω and we using Green's inequality, we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2 + \|\Delta\omega\|^2\right) + \beta_1(t)\|\omega_t\|^2 + \beta_2(t)(\varpi(x,1,t),\omega_t) = 0.$$

By multiplying the second equation in (62) by ϖ, integration over $\Omega \times (0,1)$ to get :

$$\frac{\tau}{2}\frac{d}{dt}\|\varpi\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\varpi\|^2 = 0.$$

Then

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|\varpi\|^2 d\rho + \frac{1}{2}\left(\|\varpi(x,1,t)\|^2 - \|\omega_t\|^2\right) = 0.$$

From (63), (65), we are using the Cauchy--Schwarz's inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2 + \|\Delta\omega\|^2 + \tau\int_0^1 \|\varpi\|^2 d\rho\right) + \beta_1(t)\|\omega_t\|^2 + \frac{1}{2}\|\varpi(x,1,t)\|^2$$
$$= -\beta_2(t)(\varpi(x,1,t),\omega_t) + \frac{1}{2}\|\omega_t\|^2$$
$$\le |\beta_2(t)|\|\varpi(x,1,t)\|\|\omega_t\| + \frac{1}{2}\|\omega_t\|^2.$$

Applying Young's inequality, we know that

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2+\|\Delta\omega\|^2+\tau\int_0^1\|\varpi\|^2d\rho\right)\le c\|\omega_t\|^2,$$

where c is a positive constante. Later integrate with respect to $(0,t)$, utilizing Gronwall's Lemma,

$$\|\omega_t\|^2+\|\Delta\omega\|^2+\tau\int_0^1\|\varpi\|^2d\rho=0.$$

Hence, uniqueness follows.

4 DECAY

From this point forward, we represent by c various positive constants that might vary across different instances.

Theorem 5 Assume that, f all the conditions of Theorem 3 hold. Furthermore, c,ω are positive constants:

$$E(t)\le cE(0)e^{-\omega\int_0^t\beta_1(s)ds},\quad \forall t\ge 0. \tag{50}$$

Proof We multiply by the first equation in (11) by $\phi_t E^q u$, where ϕ is a limited function that fulfills all the assumptions stated in Lemma 1. We get

$$\begin{aligned}0&=\int_S^T E^q\phi'\int_\Omega u\left(u_{tt}+\Delta^2u+\beta_1(t)u_t+\beta_2(t)z(x,1,t)\right)dxdt\\&=\left[E^q\phi'\int_\Omega uu_tdx\right]_S^T-\int_S^T\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega uu'dxdt\\&\quad-2\int_S^T E^q\phi'\int_\Omega\left(u_t^2+|\Delta u|^2\right)dxdt+\int_S^T E^q\phi'\beta_1(t)\int_\Omega uu_tdxdt\\&\quad+\int_S^T E^q\phi'\beta_1(t)\int_\Omega uu_tdxdt+\int_S^T E^q\phi'\beta_2(t)\int_\Omega uz(x,1,t)dxdt.\end{aligned}\tag{51}$$

Likewise, multiplying the second equation from (11) by $E^q\phi'\xi(t)e^{-2\tau\rho}z(x,\rho,t)$ and obtain

$$\begin{aligned}0&=\int_S^T E^q\phi'\int_\Omega\int_0^1 e^{-2\tau\rho}\xi(t)z\left(\tau z_t+z_\rho\right)dxd\rho dt\\&=\left[\frac{1}{2}E^q\phi'\xi(t)z\int_\Omega\int_0^1 e^{-2\tau\rho}z^2dxd\rho\right]_S^T-\frac{1}{2}\int_S^T\int_\Omega\int_0^1\left(E^q\phi'\xi(t)\tau e^{-2\tau\rho}\right)'z^2dxd\rho dt\\&\quad+\int_S^T E^q\phi'\int_\Omega\int_0^1\xi(t)\left(\frac{1}{2}\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2\right)+\tau e^{-2\tau\rho}z^2\right)dxd\rho dt\\&=\left[\frac{1}{2}E^q\phi'\xi(t)z\int_\Omega\int_0^1 e^{-2\pi\rho}z^2dxd\rho\right]_S^T-\frac{\tau}{2}\int_S^T\left(E^q\phi'\xi(t)\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}z^2dxd\rho dt\\&\quad+\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\left(e^{-2\tau}z^2(x,1,t)-z^2(x,0,t)\right)dxdt+\int_S^T E^q\phi'\xi(t)\tau\int_0^1\int_\Omega e^{-2\tau\rho}z^2dxd\rho dt.\end{aligned}\tag{52}$$

Summing them up,

$$
\begin{aligned}
A\int_S^T E^q\phi' dt \leq & -\left[E^q\phi'\int_\Omega uu_t dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_\Omega uu_t dxdt \\
& +2\int_S^T E^q\phi'\int_\Omega u_t^2 dxdt - \int_S^T \beta_1(t)E^q\phi'\int_\Omega uu_t dxdt \\
& -\int_S^T \beta_2(t)E^q\phi'\int_\Omega uz(x,1,t)dxdt - \left[\frac{1}{2}E^q\phi'\xi(t)\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho\right]_S^T \\
& +\frac{1}{2}\int_S^T \left(E^q\phi'\xi(t)\right)'\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho \\
& -\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\left(e^{-2\tau(t)}z^2(x,1,t) - z^2(x,0,t)\right)dxdt,
\end{aligned} \tag{53}
$$

where $A = 2\min\{1, e^{-2\tau_1}\}$. Utilizing the Cauchy-Schwarz's, Young's and Poincaré's inequalities and the fact that ϕ' is bounded and since E is an increasing function on IR^+, we arrive at the following:

$$
\left|E^q(t)\phi'\int_\Omega uu_t dx\right| \leq cE(t)^{q+1}. \tag{54}
$$

From (14), we get

$$
\begin{aligned}
\int_S^T \left|qE'E^{q-1}\phi'\int_\Omega uu_t dx\right| dt &\leq c\int_S^T E^q(t)\left|E'(t)\right| dt \leq c\int_S^T E^q(t)\left(-E'(t)\right)dt \\
&\leq c\int_S^T E^{q+1}(S),
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\int_S^T E^q\phi''\int_\Omega uu_t dxdt &\leq c\int_S^T E^{q+1}(t)\left(-\phi''\right)dt \\
&\leq cE^{q+1}(S)\int_S^T\left(-\phi''\right)dt \leq cE^{q+1}(S),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_S^T E^q(t)\phi'\int_\Omega u_t^2 dxdt &\leq c\int_S^T E^q\phi'\frac{1}{\beta_1(t)}\int_\Omega \beta_1(t)u_t^2 dxdt \\
&\leq \int_S^T E^q\frac{\phi'}{\beta_1(t)}\left(-E'\right)dt.
\end{aligned} \tag{56}
$$

Define

$$
\phi(t) = \int_0^t \beta_1(s)ds. \tag{57}
$$

Now assume that ϕ is the increasing concave function of class C^1 over IR^+, it is limited and

$$
\phi(t) \to +\infty \quad as \quad t \to +\infty. \tag{58}
$$

Thus, we conclude, by (56), that

$$
\int_S^T E^q\phi'\int_\Omega u_t^2 dxdt \leq c\int_S^T E^q\left(-E'\right)dt \leq cE^{q+1}(S), \tag{59}
$$

From the (**H1**) assumption, Young's and Poincaré's inequality and (14),

$$
\begin{aligned}
\left|\int_S^T E^q\phi'\int_\Omega uu_t dxdt\right| &\leq c\int_S^T E^q\phi'\|u\|\|u_t\|dt \\
&\leq c\varepsilon'\int_S^T E^q\phi'\|u\|^2 dt + c(\varepsilon')\int_S^T E^q\phi'\|u_t\|^2 dt \\
&\leq \varepsilon' c_*\int_S^T E^q\phi'\|\Delta u\|^2 dt + c(\varepsilon')\int_S^T E^q\phi'\|u_t\|^2 dt \\
&\leq \varepsilon' c_*\int_S^T E^{q+1}\phi' dt + cE^{q+1}(S).
\end{aligned}
$$

Remembering that $\xi'\leq 0$ and the definition of E, selecting ε' to be sufficiently small we can

$$
\begin{aligned}
\int_S^T \left(E^q\xi(t)\right)'\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho dt &\leq \int_S^T \left(E^q\right)'\xi(t)\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2 dxd\rho dt \\
&\leq c\int_S^T E^q|E'|dt \\
&\leq c\int_S^T E^q(-E'(t))dt \\
&\leq cE^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T E^q\xi(t)\int_\Omega e^{-2\tau}z^2(x,1,t)dxdt &\leq c\int_S^T E^q\xi(t)\int_\Omega z^2(x,1,t)dxdt \\
&\leq c\int_S^T E^q(-E')dt \\
&\leq cE^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T E^q\xi(t)\int_\Omega z^2(x,0,t)dxdt &= \int_S^T E^q\xi(t)\int_\Omega u_t^2(x,t)dxdt \\
&\leq cE^{q+1}(S),
\end{aligned}
$$

from (53) and the previous estimates we obtain

$$
\int_S^T E^q\phi' dt \leq CE^{q+1}(S).
$$

We get the property of the exponential stability of the solution.

REFERENCES:

1. Adams R.A., Fournier J.J.F., Sobolev Spaces, Academic Press, New York, 2003.
2. Al-Gharabli M. M. and Al- Mahdi M., Existence and stability results of a plate equation with nonlinear damping and source term, Electronic Research Archive, 30 (11) (2022), 4038-4065.
3. Benaissa A., Benguessoum A. and Messaoudi S. A., Energy decay of solutions for wave equation with a constant weak delay and a weak internal feedback, Electronic Journal of Qualitative Theory of Differential Equations, (2014), 11, 1-13.
4. Fidan A., Pişkin E. and Çelik E. Existence, Decay, and Blow-up of Solutions for a Weighted m-Biharmonic Equation with Nonlinear Damping and Source Terms. Journal of Function Spaces, 2024(1), 5866792.
5. Freitas P., Zuazua E., Stability results for the wave equation with indefinite damping, Journal of Differential Equations, 132 (2) (1996), 338-352.
6. Fridman E., Introduction to time-delay systems*:* Analysis and control. Springer.(2014).

7. Georgiev V., Todorova G., Existence of a solution of the wave equation with nonlinear damping and source terms, Journal of Differential Equations, 109 (2) (1994) 295-308.
8. Guo B., Li X., Bounds for the lifespan of solutions to fourth-order hyperbolic equations with initial data at arbitrary energy level, Taiwanese Journal of Mathematics, (2019), 23: 1461-1477.
9. Han Y., Blow-up phenomena for a reaction diffusion equation with special diffusion process. Applicable Analysis, 101(6) (2022), 1971-1983.
10. Lions J.L., Quelques methodes de resolution des problemes aux limites non lineaires, Dunod, Paris 1969.
11. Kafini M. and Messaoudi S A., A blow-up result in a nonlinear wave equation with delay, Mediterranean Journal of Mathematics, 13 (2016) 237--247.
12. Martinez P., A new method to obtain decay rate estimates for dissipative systems, ESAIM Control, Optimisation and Calculus of Variations, 4**(**1999**),** 419-444.
13. Meyvacı M., Decay Estimate for the Time-Delayed Fourth- Order Wave Equations, Mathematical Sciences and Applications E-Notes, 9(3) (2021), 142-150.
14. Nicaise S., Pignotti C., Stability and instability results of the wave equation with delay term in the boundary or internal feedbacks, SIAM Journal on Control and Optimization, 45(2006), 1561--1585.
15. Pişkin E., Okutmuştur B., An Introduction to Sobolev Spaces, Bentham Science, 2021.
16. Pişkin E., Fidan A., Nonexistence of global solutions for the strongly damped wave equation with variable coefficients. Universal Journal of Mathematics and Applications, 5(2) (2022), 51-56.
17. Pişkin E., Fidan A., Finite time blow up of solutions for the m-Laplacian equation with variable coefficients. Al-Qadisiyah Journal of Pure Science, 28:1 (2023).
18. Wu S., Lower and Upper Bounds for the Blow- Up Time of a Class of Damped Fourth-Order Nonlinear Evolution Equations, Springer, 24 (2018) 287-295.
19. Zheng X., Shang Y., Peng X., Blow up of Solutions for a Nonlinear Petrovsky Type Equation with Time-dependent Coeffcients, Acta Mathematicae Applicatae Sinica, English Series, 36(4) (2020) 836-846.