ON THE MATRIX REPRESENTATION OF GENERALIZED FIBONACCI SEQUENCES AND APPLICATIONS

K. L. Verma

Department of Mathematics
Career Point University, Hamirpur (HP) 176041, INDIA
Corresponding Author Email: klverma@netscape.net, klverma@cpuh.edu.in

Abstract

In this paper, we have developed the matrix representation of the (p,q)-generalized Fibonacci sequence by expressing the homogeneous linear recurrence relations $V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}$ as $V_n(p,q,a,b) = T_{n-1}(p,q)V_2 + qT_{n-2}(p,q)V_1$, $\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, n \geq 3\ T_0, = 0, T_1 = p, T_2 = 1.$ $p, q, V_1 = a$ and $V_2 = b$ are arbitrary integers. Employing these, sequence is characterized in the form of generalized matrix, which is used to investigate the new results and generalized the exiting results, reducing them as special cases. Selecting any suitable values of (p,q) initial terms and *n*, general matrix is utilized for encryption of data in the form of second order matrix.

Keywords: *Recurrence relations, generalized, Fibonacci sequence, encryption,*

INTRODUCTION

Fibonacci sequences are the legendary numerical sequences in mathematical sciences, having imperious and exciting applications in physical and computational sciences (see Gulec and Taskarar (2009), Koshy (2011) and Kilic (2007)).Authors Silvester (1979), Stakhov (1999) and Adam (2016) have employed the matrix representations of the Fibonacci sequences to examine the properties of sequences and investigated the power of the second order matrices, resulted from the generalizations of the Fibonacci sequences.

The Fibonacci number sequence fascinates the curiosity of scientists across all disciplines. Viliam et al. (2019) considered the Fibonacci numbers for using the computing environment to study some practical applications. Matoušová, et al. (2020) employed (2, q)-distance Fibonacci numbers for algorithms development. Meng et al.(2023) studied the practical applications of revisiting the spiral pattern in rib stiffening design using such sequences. Yilmaz et al. (2024) studied the basic properties by introducing distance Fibonacci sequences generalization.

Authors Basu and Prasad (2009), Stakhov (2006) considered square matrices of the classical Fibonacci numbers to established the generalized relations and their applications. Kalman and Mena (2009) and Tran (2024) studied several well-known fascinating identities.

In this article, firstly (p,q) generalized Fibonacci sequences is defined in two different ways and established in the form of generalized square 2 × 2 matrix with all the elements are in the general form. Some results are defined and proved in the comprehensive form without losing the generality. Matrix representation of the generalized Fibonacci sequences is presented, and then properties of this matrix are studied in the general form. It is also shown that on substituting particular initial values and coefficients $V_1 = a, V_2 = b$ and p,q of the recurrence relations, various well known results like classical Fibonacci and Lucas sequences etc. become special cases of this generalization. The result obtained are utilized to study the various results identities and considered some applications. Some existing theorems are deduced as special cases by specializing the (p, q) and initial terms.

Special Cases

When the initial values are $a=0, b=1$, and $p=q=1$ then the resulting sequence is the *classical Fibonacci sequence.*
When $a=2, b=1,$ and $p=q=1$ then the (p,q) sequence is the *Lucas sequence*. We are considering $V_n(p,q,a,b)$ in short form as V_n.

Theorem 1 Let $V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}$, $n \geq 3, V_1 = a,\ V_2 = b$.
then $V_n = T_{n-1}V_2 + qT_{n-2}V_1,\ n \geq 3$, where

$$\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, n > 2, T_1 = 1,\ T_2 = p .$$

Proof: We shall prove the theorem using induction on $n \geq 3$.

Firstly, we shall verify the validity of the theorem for $n = 3, 4$.
When $n = 3$, we have, LHS:

$$V_3 = pV_2 + qV_1 = pb + qa \quad \text{(by definition)}$$

and RHS

$$V_3 = T_2V_2 + qV_1 = pb + qa$$

$$V_n = T_{n-1}V_2 + qT_{n-2}V_1,\ n \geq 3$$

Similarly for $n = 4$ we have

$$V_4 = pV_3 + qV_2 = p(pb + qa) + qb = (p^2 + q)b + pqa \quad \text{(by definition)}$$

and

$$V_4 = T_3V_2 + qT_2V_1 = (p^2 + q)b + pqa = p(p^2 + 2q)b + q(p^2 + q)a$$

Thus the theorem is true for $n = 3, 4$.
Assuming that it is valid for $n = k$ (k is greater than > 4).
Therefore

$$V_k = T_{k-1}V_2 + qT_{k-2}V_1,\ k \geq 4 \qquad (1)$$

Now we shall show that theorem 1 is valid for $n = k+1$ when it is valid for $n = k$.
Multiply p and then adding term qV_{k-1} both sides of (1), we have

$$pV_k + qV_{k-1} = p\left(T_{k-1}V_2 + qT_{k-2}V_1\right) + q\left(T_{k-2}V_2 + qT_{k-3}V_1\right)$$
$$= \left(pT_{k-1} + qT_{k-2}\right)V_2 + q\left(T_{k-2} + qT_{k-3}\right)V_1$$
$$= T_kV_2 + qT_{k-1}V_1$$
$$\left(\because\ T_k = pT_{k-1} + qT_{k-2}, T_{k-1} = pT_{k-2} + qT_{k-3}\right)$$

Using definition to the left hand side and after simplifying the right hand side of the above equation, we obtain

$$V_{k+1} = T_kV_2 + qT_{k-1}V_1,\ k \geq 3$$

.

Hence the result follows from induction.

Theorem 2. For $n > 0$,

$$V_{n+6} = p\left(p^2 + 3q\right)V_{n+3} + q^3V_n,$$

Proof.

We shall prove the theorem using induction on *n*.

For $n = 1$, we have $V_7 = p\left(p^2 + 3q\right)V_4 + q^3V_1,$.

Using the definition, we obtain

$$V_7 = \left(p^6 + 5p^4q + 6q^2p^2 + q^3\right)V_1 + q\left(p^5 + 4p^3q + 3q^2p\right)V_2.$$

Also

$$V_7 = p\left(p^2 + 3q\right)V_4 + q^3V_1$$
$$= p\left(p^2 + 3q\right)\left[\left(p^2q + q^2\right)V_1 + \left(p^3 + 2pq\right)V_2\right] + q^3V_1$$
$$= \left(p^6 + 5p^4q + 6q^2p^2 + q^3\right)V_1 + q\left(p^5 + 4p^3q + 3q^2p\right)V_2$$

Thus result is valid for $n = 1$.

Now assume it is valid for $n > k$. ($k > 1$), that is

$$V_{k+6} = p\left(p^2 + 3q\right)V_{k+3} + q^3V_k, \tag{2}$$

Now it will be shown that the result is valid for $n = k + 1$.

Now from (2), we have $V_{(k+1)+6} = p\left(p^2 + 3q\right)V_{(k+1)+3} + q^3V_{(k+1)},$

$$= p\left(p^2 + 3q\right)V_{k+4} + q^3V_{k+1},.$$

Using definition of $V_n = pV_{n-1} + qV_{n-2}$, $n > 3$ we have

$$V_{(k+1)+6} = p\left(p^2 + 3q\right)V_{k+4} + q^3V_{k+1},$$
$$= p\left(p^2 + 3q\right)\left(pV_{k+3} + qV_{k+2}\right) + q^3\left(pV_k + qV_{k-1}\right)$$
$$= p\left(p\left(p^2 + 3q\right)V_{k+3} + q^3V_k\right) + q\left(p\left(p^2 + 3q\right)V_{k+2} + q^3V_{k-1}\right)$$
$$= pV_{k+6} + qV_{k+5}$$

Using the assumption that

$$\left[\because V_{k+6} = \left(p\left(p^2+3q\right)V_{k+3} + q^3V_k\right), V_{k+5} = \left(p\left(p^2+3q\right)V_{k+2} + q^3V_{k-1}\right)\right]$$

Hence the theorem is true for $n = k + 1$. Therefore, the result is true for every $n > 0$

$$\begin{aligned} V_{(k+1)+6} &= p\left(p^2+3q\right)V_{k+4} + q^3V_{k+1}, \\ &= p\left(p^2+3q\right)\left(pV_{k+3} + qV_{k+2}\right) + q^3\left(pV_k + qV_{k-1}\right) \\ &= p\left(p\left(p^2+3q\right)V_{k+3} + q^3V_k\right) + q\left(p\left(p^2+3q\right)V_{k+2} + q^3V_{k-1}\right). \\ &= pV_{k+6} + qV_{k+5} \end{aligned}$$

$$\left(\because V_n = pV_{n-1} + qV_{n-2}\ , n > 3\right)$$

Remark: The above identity can be put in another form

$$V_{n+6} = \left(p^5 + 4p^3q + 3pq^2\right)V_{n+1} + \left(p^4q + 3p^2q^2 + q^3\right)V_n$$

$$V_{n+6} = \left(p^4 + 3p^2q + q^2\right)V_{n+2} + q\left(p^3 + 2pq\right)V_{n+1}$$

$$V_{n+6} = pV_{n+5} + qV_{n+4}$$

MATRIX REPRESENTATION OF GENERALIZED FIBONACCI SEQUENCES

Theorem 3.

If $M = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$, then $M^n = \begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, n > 2, T_1 = 1,\ T_2 = p.$$

Proof

We shall prove the result, using induction on n

$$M^1 = \begin{pmatrix} T_2 & qT_1 \\ T_1 & qT_0 \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} =, M^2 = \begin{pmatrix} T_3 & qT_2 \\ T_2 & qT_1 \end{pmatrix} = \begin{pmatrix} p^2+q & pq \\ p & q \end{pmatrix}$$

The result is true for $n = 1$ and 2.

Assume that the result is valid for $n = m$. Then

$$M^m = \begin{pmatrix} T_{m+1} & qT_m \\ T_m & qT_{m-1} \end{pmatrix} \tag{3}$$

Now we shall show that theorem is also valid for $n = m + 1$ when it is valid for $n = m$

$$M^{m+1} = M^m M = \begin{pmatrix} T_{m+1} & qT_m \\ T_m & qT_{m-1} \end{pmatrix} \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} pT_{m+1} + qT_m & qT_{m+1} \\ pT_m + qT_{m-1} & qT_m \end{pmatrix}$$
$$= \begin{pmatrix} T_{m+2} & qT_{m+1} \\ T_{m+1} & qT_m \end{pmatrix}$$

Now using $T_{m+2} = pT_{m+1} + qT_m$, $T_{m+1} = pT_m + qT_{m-1}$

Hence the theorem follows by mathematical induction.

Theorem 4

If $M^n = \begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$, then

$$\det\left(M^n\right) = q\left(T_{n+1}T_{n-1} - T_n^2\right) = \left(-q\right)^n$$

Proof:

To find the determinant of matrix M , it is calculated as

For n= 1, $\det\left(M\right) = q\left(T_2T_0 - T_1^2\right) = q\left(0-1\right) = \left(-1\right)^1 q^1$

Now using the property of determinant of the product of two square matrices, we have

$$\det\left(M^k\right) = \det\left(M\right)^k = \left[q\left(T_2T_0 - T_1^2\right)\right]^k = \left(-q\right)^k.$$

Thus we have

$$\det\left(M^n\right) = \left[q\left(T_2T_0 - T_1^2\right)\right]^n = \left(-q\right)^n$$

which prove the theorem.

Corollary

Using the definition of M and M^n, we have

$$M^{n+1}M^n = \begin{pmatrix} T_{2n+2} & qT_{2n+1} \\ T_{2n+1} & qT_{2n} \end{pmatrix} = M^{2n+1}$$

Theorem 5.

If $M^n = \begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$, then $M^n = pM^{n-1} + qM^{n-2}$.

Proof: Since

$$M^n = \begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$$

$$\left(\because \quad T_{n+1} = pT_n + qT_{n-1}, T_n = pT_{n-1} + qT_{n-2}, T_{n-1} = pT_{n-2} + qT_{n-3} \right)$$

$$\therefore M^n = \begin{pmatrix} pT_n + qT_{n-1} & q\left(pT_{n-1} + qT_{n-2}\right) \\ pT_{n-1} + qT_{n-2} & q\left(pT_{n-2} + qT_{n-3}\right) \end{pmatrix}$$

$$= \begin{pmatrix} pT_n & qpT_{n-1} \\ pT_{n-1} & qpT_{n-2} \end{pmatrix} + \begin{pmatrix} qT_{n-1} & qqT_{n-2} \\ qT_{n-2} & qqT_{n-3} \end{pmatrix}$$

$$= p\begin{pmatrix} T_n & qT_{n-1} \\ T_{n-1} & qT_{n-2} \end{pmatrix} + q\begin{pmatrix} T_{n-1} & qT_{n-2} \\ T_{n-2} & qT_{n-3} \end{pmatrix}$$

$$M^n = pM^{n-1} + qM^{n-2}$$

.

Corollary:

$$qM^{n-2} = M^n - pM^{n-1}.$$

Theorem 5 For $M^n = \begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$,

$$M^{n+m} = M^n M^m = M^m M^n.$$

Proof: From the previous theorem we have

$$M^n = pM^{n-1} + qM^{n-2}$$

$$M^m = pM^{m-1} + qM^{m-2}$$

This implies

$$M^{n+m} = pM^{n+m-1} + qM^{n+m-2}$$

$$M^{n+m} = M^n M^m = M^m M^n$$

$$M^{n+m} = \begin{pmatrix} T_{n+m+1} & qT_{n+m} \\ T_{n+m} & qT_{n+m-1} \end{pmatrix}$$

$$T_{m+2} = pT_{m+1} + qT_m,\ T_{m+1} = pT_m + qT_{m-1}$$

Result follows.

INVERSE OF THE GENERALIZED FIBONACCI MATRIX M^n

Theorem 6. For $M = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$**,** then

$$M^{-n} = \frac{1}{(-1)^n q^{n-1}} \begin{pmatrix} T_{n-1} & -T_n \\ -T_n / q & T_{n+1} / q \end{pmatrix}, \text{ where}$$

$$\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, n > 2, T_2 = p, T_1 = 1, T_0 = 0 \quad T_0 = 0$$

.,

Proof: Using induction, we have,

$$M^{-1} = \frac{1}{-q}\begin{pmatrix} 0 & -q \\ -1 & p \end{pmatrix} = \frac{1}{(-)^1 q^{1-1}}\begin{pmatrix} T_{1-1} & -T_1 \\ -T_1/q & T_2/q \end{pmatrix}, \left[= \frac{1}{(-)^1 q^{1-1}}\begin{pmatrix} 0 & -1 \\ -1/q & p/q \end{pmatrix}\right],$$

$$M^{-2} = \frac{1}{q^2}\begin{pmatrix} q & -pq \\ -p & p^2+q \end{pmatrix} = \frac{1}{(-1)^2 q^{2-1}}\begin{pmatrix} T_1 & -T_2 \\ -T_2/q & T_3/q \end{pmatrix}$$

$$= \frac{1}{q}\begin{pmatrix} 1 & -p \\ -p/q & \left(p^2+q\right)/q \end{pmatrix}$$

$$= \frac{1}{q^2}\begin{pmatrix} q & -pq \\ -p & \left(p^2+q\right) \end{pmatrix}.$$

The result is true for $n = 1$ and 2.

Assume that the theorem is valid for $n = k$

Now we shall show that theorem is also valid for $n = k+1$ when it is valid for $n = k$.

$$M^{-k} = \frac{1}{(-1)^k q^{k-1}}\begin{pmatrix} T_{k-1} & -T_k \\ -T_k/q & T_{k+1}/q \end{pmatrix}.$$

Now for $n = k+1$ we have

$$M^{-(k+1)} = \frac{1}{(-1)^{k+1} q^k}\begin{pmatrix} T_k & -T_{k+1} \\ -T_{k+1}/q & T_{k+2}/q \end{pmatrix}.$$

$$M^{-(k+1)} = M^{-k}M^{-1}$$

$$= \frac{1}{(-1)^k q^{k-1}}\begin{pmatrix} T_{k-1} & -T_k \\ -T_k/q & T_{k+1}/q \end{pmatrix}\left[\frac{1}{(-)^1 q^{1-1}}\begin{pmatrix} 0 & -1 \\ -1/q & p/q \end{pmatrix}\right].$$

$$= \frac{1}{(-1)^{k+1} q^{k-1}}\begin{pmatrix} T_k/q & -T_{k-1} - pT_k/q \\ -T_{k+1}/q^2 & T_k/q + pT_{k+1}/q^2 \end{pmatrix}.$$

$$= \frac{1}{(-1)^{k+1} q^k}\begin{pmatrix} T_k & -(qT_{k-1} + pT_k) \\ -T_{k+1}/q & \left(qT_k + pT_{k+1}\right)/q \end{pmatrix}.$$

$$= \frac{1}{(-1)^{k+1} q^k}\begin{pmatrix} T_k & -T_{k+1} \\ -T_{k+1}/q & T_{k+2}/q \end{pmatrix}$$

$$\because qT_{k-1} + pT_k = T_{k+1}, qT_k + pT_{k+1} = T_{k+2}.$$

Hence by induction result is true.

APPLICATION OF THE GENERALIZED MATRIX M^n FOR CODING AND DECODING

In this section, Generalized Fibonacci matrix M^n (for any coefficients p and q of second order recurrence relation and arbitrary initial terms V_1 and V_2 is defined for any four digit numbers, which is taken as a 4-digit initial code represented in a 2×2 square matrix.
For coding generalized Fibonacci matrix M^n of order 2 and for decoding its inverse $\left(M^{-n}\right)$ are considered.
Since

$$M=\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

and

$$M^n=\begin{pmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{pmatrix}$$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix}=\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} p \\ 1 \end{pmatrix}, n>2, T_2=p, T_1=1, T_0=0$$

Assume that original message is in the form of a square matrix of order 2, is

$$Q=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, m_i\geq 0.$$

Now for coding, choose any value of n, say, $n=4$, then we have M^4

$$M^4=\begin{pmatrix} T_5 & qT_4 \\ T_4 & qT_3 \end{pmatrix}=\begin{pmatrix} \left[\left(p^2+q\right)p+pq\right]p+\left(p^2+q\right)q & q\left(p^2+q\right)p+pq \\ \left(p^2+q\right)p+pq & q\left(p^2+q\right) \end{pmatrix}$$

Now find inverse of M^{-4}, which is

$$M^{-4}=\frac{1}{(-)^4 q^{4-1}}\begin{pmatrix} T_3 & -T_4 \\ -T_4/q & T_5/q \end{pmatrix}$$

$$=\frac{1}{q^3}\begin{pmatrix} p^2+q & -\left(p^2+q\right)p+pq \\ -\left[\left(p^2+q\right)p+pq\right]/q & \left[\left[\left(p^2+q\right)p+pq\right]p+\left(p^2+q\right)q\right]/q \end{pmatrix}$$

Now, let the product of the matrices QM^4 is denoted by matrix R, which is

$$QM^4=\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}=R$$

where

$$\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} \left[\left(p^2+q\right)p+pq\right]p+\left(p^2+q\right)q & q\left(p^2+q\right)p+pq \\ \left(p^2+q\right)p+pq & q\left(p^2+q\right) \end{pmatrix}$$

The code matrix R is then sent, decoding is then done using the matrix equation

$$RM^{-4} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = Q.$$

Determinant of the code matrix R, when q is arbitrary

$$.\det\left(QM^4\right) = \det(Q)\det\left(M^4\right) = (-q)^n \det(Q)$$

If $q = 1$, then

$$\det\left(QM^4\right) = \det(Q)\det\left(M^4\right) = (-1)^n \det(Q).$$

If $p = 1$ and $q = 1$, then

$$\det\left(QM^4\right) = \det(Q)\det\left(M^4\right) = \det(Q).$$

NUMERICAL RESULTS AND DISCUSSION:

Example. First of all choose any p and q for the Generalized Fibonacci matrix and any four digit number for messaging. Here we are choosing $p = 2$ and $q = 3$ and the original message 1956 in the matrix form

$$Q = \begin{pmatrix} 1 & 9 \\ 5 & 6 \end{pmatrix}.$$

Now the Generalized Fibonacci M matrix, when $n = 4$ as a coding matrix

$$M^4 = \begin{pmatrix} T_5 & qT_4 \\ T_4 & qT_3 \end{pmatrix}$$

$$= \begin{pmatrix} \left[\left(p^2+q\right)p + pq\right]p + \left(p^2+q\right)q & q\left(p^2+q\right)p + pq \\ \left(p^2+q\right)p + pq & q\left(p^2+q\right) \end{pmatrix} = \begin{pmatrix} 61 & 60 \\ 20 & 21 \end{pmatrix}.$$

Now the message forwarded in the form of matrix is

$$R = QM^4 = \begin{pmatrix} 241 & 249 \\ 425 & 426 \end{pmatrix} = \left[\because \begin{pmatrix} 61 & 60 \\ 20 & 21 \end{pmatrix}\begin{pmatrix} 1 & 9 \\ 5 & 6 \end{pmatrix} = R\right].$$

Now to decode the message in the form of matrix, we have

$$RM^{-4} = \begin{pmatrix} 241 & 249 \\ 425 & 426 \end{pmatrix}\begin{pmatrix} 7/27 & -20/27 \\ -20/81 & 61/81 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 5 & 6 \end{pmatrix} = Q.$$

In this way, we can take the different p, q and n, same message can be send differently, like CAPTCHA, which one cannot when p and q are fixed.

CONCLUSION

Configurations are important in developing results for sequences in the most generalized forms. On employing such configurations, a generalized procedure was obtained in solving nth term of Fibonacci-like sequence using its first two terms and the coefficients of the linear recurrence relations and generalized Fibonacci numbers. The procedure will deliver amazingly support for other researchers to establish new theorems, identities and applications in mathematical sciences.

ACKNOWLEDGEMENT

The authors are very grateful to the referee for constructive suggestions and remarks which resulted in improving the quality and presentation of the paper.

REFERENCES:

1. Silvester, J. R.: " Fibonacci properties by matrix methods". The Mathematical Gazette , 63(425), (1979). 188–191.

2. Gulec H. H. and Taskarar N.: "On the Properties of Fibonacci Numbers with Binomial Coefficients". Int. J. Contemp. Math. Sciences, **4(25),** , (2009), 1251 –1256. doi:10.2307/3617892.

3. Koshy.T.: Fibonacci and Lucas Numbers with Applications, John Wiley Sons, Netherlands, (2011).

4. Kilic, E.: "Sums Of Generalized Fibonacci Numbers By Matrix Methods". Ars Combinatoria, **84**: , (2007), 23–31.

5. Basu, M, and Prasad, B.: "The generalized relations among the code elements for Fibonacci coding theory". Chaos, Solitons and Fractals, **41(5),** (2009), 2517 –2525. doi:10.1016/j.chaos.2008.09.030.

6. Adam, M.: "Powers of the generalized 2-Fibonacci matrices". Journal of Applied Mathematics Bioinformatics, **6(3),** (2016), 145–154.

7. Stakhov, A. P. "A generalization of the Fibonacci Q-matrix". Rep. Nat. Acad. Sci.Ukraine, **9**, (1999), 46–49.

8. Stakhov, A. P.: " Fibonacci matrices, a generalization of the Cassini formula, and a new coding theory". Chaos, Solitons Fractals, **30(1),** (2006), 55–66.doi:10.1016/j.chaos.2005.12.054.

9. Kalman, D. and Mena, R.: "The Fibonacci Numbers Exposed". Mathematics Magazine , **76(3),** , (2003), 167–181. doi: 10.1080/0025570X.2003.11953176.

10. Viliam Ďuriš, Lucia Rumanová , Dusan Vallo and Júlia Záhorská, "Fibonacci Numbers and Selected Practical Applications in theMatlab Computing Environment", Acta Mathematica Nitriensia **5(1),**.(2019), 14 – 22. doi: 10.17846/AMN.2019.5.1.14-22

11. Matoušová, I.and Trojovský, P., "On Coding by (2, q)-Distance Fibonacci numbers," Mathematics, MDPI, 8(11), (2020) 1-24. **doi.org/10.3390/math8112058**

12. Meng, L.; Zhang, J.; Hou, Y.; Breitkopf, P.; Zhu, J.; Zhang, W. Revisiting the Fibonacci Spiral Pattern for Stiffening Rib Design. Int. J. Mech. Sci. 246(2023), 108131. https://doi.org/10.1016/j.ijmecsci.2023.108131

13. Tran, Quang Hung "A New Improvement of a Fibonacci Inequality," Journal of the South Carolina Academy of Science: Vol. 21(3), (2024), 90-92. https://scholarcommons.sc.edu/jscas/vol21/iss3/11

14. Yilmaz, N.Ş.; Włoch, A. Özkan, E.” Generalization of the Distance Fibonacci Sequences”, Axioms MDPI, 13(2024) 420. doi.org/10.3390/axioms13070420.