Evaluating Life Time Models: A Comparative Study of the Inverse Weibull and Inverse Lognormal Distributions

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Abstract

This study presents a comparative analysis of the Inverse Weibull and Inverse Lognormal distributions using both simulated and real-world data with Maximum Likelihood Estimation (MLE). Model selection criteria included Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (AD), and Kolmogorov-Smirnov (KS) tests. Six simulated sample sizes (50, 100, 150, 200, 250, and 500) were used, with 1000 replications each. Results showed the Inverse Lognormal distribution consistently had lower AIC and BIC values at small to moderate sample sizes. Furthermore, real-world stock price data (sample size 100) from the Nigerian Stock Exchange was analyzed. Descriptive statistics and goodness-of-fit tests favored the Inverse Lognormal model. These findings support the utility of AIC, BIC, AD, and KS in lifetime data modeling and model selection.

Keywords: *Inverse Weibull distribution; Inverse Lognormal distribution; Maximum Likelihood Estimation; AIC; BIC; Anderson-Darling; Kolmogorov-Smirnov; Simulation study; Lifetime analysis; Model selection; Stock price data*

1. Introduction

In recent decades, the analysis of lifetime and reliability data has gained substantial traction in both theoretical and applied statistics. Fields such as biostatistics, engineering, environmental sciences, and actuarial modeling rely heavily on accurate lifetime models to inform decisions and policies. Distributions that can effectively capture the skewness and tail behavior of real-world data are therefore of critical importance. Among such models, the Inverse Weibull (IW) and Inverse Lognormal (ILN) distributions stand out due to their versatility and ability to handle decreasing hazard rates (Barreto-Souza and Cribari-Neto, 2009).

The Inverse Weibull distribution was originally proposed by Keller and Kamath (1982) and has since found applications in medical survival analysis, reliability engineering, and biological modeling. It has proven especially useful for systems that exhibit early-life failure patterns or burn-in periods, such as electronic devices or biological specimens (Cooray, 2006; Lemonte and Cordeiro, 2013). This distribution is known for its flexibility, defined by a shape and scale parameter that allow it to model a wide variety of real-life scenarios involving heavy tails and negative skewness.

In parallel, the Inverse Lognormal distribution is rooted in the classical lognormal model but is adapted to handle scenarios where the reciprocal of a lognormally distributed variable is more meaningful. It has been widely applied in finance to model inverse asset returns, in environmental studies for pollutant concentrations, and in hydrology to represent rainfall and drought characteristics (Crow and Shimizu, 1988; Tahmasbi and Rezaei, 2008; Singh et al., 2012). Its mathematical tractability and positive support make it an attractive alternative for modeling skewed data.

Despite their individual strengths, there remains an ongoing debate in the statistical literature regarding which of these two distributions performs better under varying conditions. Model selection is especially critical when the chosen distribution is to be used for risk assessment or policy implications. Traditional goodness-of-fit tests such as the Anderson-Darling (Stephens, 1974) and Kolmogorov-Smirnov (Thode, 2002) remain commonly used tools. However, these are often limited in their power to discriminate between close-fitting models, especially in small sample settings. To mitigate these limitations, information-based selection criteria like the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are preferred. These criteria account for both model complexity and likelihood, offering a more nuanced model comparison (Burnham and Anderson, 2002).

Numerous studies have explored the statistical properties of these models. For example, Kundu and Raqab (2005) examined the efficacy of MLE for IW parameters under censored data, while Alizadeh and Zakerzadeh (2018) proposed a generalized IW model suitable for Type-II censoring. Similarly, Bourguignon et al. (2014) analyzed the flexibility of the ILN model for modeling heavy-tailed data, and Wu and Tsai (2007) compared it with other inverse models under various estimation techniques. Furthermore, Okorie and Ukoje (2021) demonstrated the practical utility of AIC and BIC in model validation in biomedical studies, and Oduaran et al. (2022) stressed the importance of model selection in financial analytics using stock return distributions.

Simulation-based comparisons offer a rigorous platform for evaluating model performance under controlled conditions. Monte Carlo simulations have become an indispensable technique in assessing statistical estimators and selection procedures. As emphasized by Hossain and Ali (2020), simulations allow researchers to systematically vary sample sizes and model parameters to uncover performance patterns. They are particularly helpful in identifying scenarios where one distribution may outperform another in terms of parsimony and predictive power.

In this study, we bridge the theoretical and empirical gap by providing both a comprehensive simulation study and a real-world application. We generate random samples from IW and ILN distributions with known parameters and compare their performance across six sample sizes (n = 50, 100, 150, 200, 250, and 500), each replicated 1000 times. The fit of each model is evaluated using AIC, BIC, Anderson-Darling, and Kolmogorov-Smirnov statistics. To validate the simulation findings, we further apply both models to actual financial data obtained from the Nigerian Stock Exchange.

The contributions of this study are threefold: (i) the extensive application of Monte Carlo simulations across a wide range of sample sizes, (ii) a comprehensive model comparison using both classical and modern information criteria, and (iii) the incorporation of real-world financial data to validate theoretical results. These contributions not only enhance our understanding of inverse distributions but also provide practitioners with practical tools for model selection in lifetime and financial data analysis.

The remainder of this article is structured as follows: Section 2 presents the methodology and MLE derivations for both models. Section 3 discusses the simulation setup and the model selection criteria. Section 4 contains the results of the simulation and real-world analyses. Section 5 offers conclusions and outlines future directions.

2. Methodology

In this section , we derive the Maximum Likelihood Estimate for the parameters of Inverse Weibull (IW) and Inverse Lognormal (ILN) distributions and describe the simulation framework used to compare them.

2.1 Inverse Weibull Distribution

Let $X : IW(\alpha, \beta)$ where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. The probability density function (PDF) of the Inverse Weibull Distribution is

$$f(x, \alpha, \beta) = \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{-\alpha-1} \exp\left\{-\left(\frac{x_i}{\beta}\right)^{-\alpha}\right\}, x > 0 \tag{1}$$

Given a random sample of $x_1, x_2, ..., x_n$ from this distribution, the log-likelihood function is

$$l(\alpha, \beta) = n\log(\alpha) - n\log(\beta) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} \tag{2}$$

To obtain the Maximum Likelihood Estimate of $\hat{\alpha}$ and $\hat{\beta}$, we take partial derivatives of Equation (2) with respect to α and β and set them to zero, then ,we have

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}\log\left(\frac{x_i}{\beta}\right) = 0 \tag{3}$$

$$\frac{\partial \mathrm{l}}{\partial \beta}=\frac{n\alpha}{\beta}+\alpha\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}\left(\frac{x_i}{\beta^2}\right)=0 \tag{4}$$

These equations are linear and require numerical methods like Newton-Raphson to solve for Maximum Likelihood Estimates

2.2 Inverse Lognormal Distribution

Let X follows Inverse Lognormal Distribution which follows that

$X=\frac{1}{Y}$, where $Y: \ Lognormal\left(\mu,\sigma^2\right)$

The Probability Density Function of the Inverse Lognormal is given by

$$f\left(x;\mu,\sigma\right)=\frac{1}{x^2\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(\log\frac{1}{x}-\mu\right)^2}{2\sigma^2}\right), x>0 \tag{5}$$

Equation (5) simplifies to

$$f\left(x;\mu,\sigma\right)=\frac{1}{x^2\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(-\left(\log x\right)-\mu\right)^2}{2\sigma^2}\right) \tag{6}$$

The log-likelihood function for a random sample $x_1,x_2,...,x_n$ is given by

$$\mathrm{l}\left(\mu,\sigma\right)=-n\log\left(\sigma\right)-2\sum_{i=1}^{n}\log\left(x_i\right)-\frac{n}{2}\log\left(2\pi\right)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\log\left(\frac{1}{x_i}\right)-\mu\right)^2 \tag{7}$$

Taking the partial derivatives of equation (7) with respect to μ and σ and setting them to zero, then we have

$$\frac{\partial \mathrm{l}}{\partial \mu}=\frac{1}{\sigma^2}-\sum_{i=1}^{n}\left(\log\left(x_i\right)-\mu\right)=0$$

$$\Rightarrow \hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{1}{x_i}\right) \tag{8}$$

$$\frac{\partial \mathrm{l}}{\partial \sigma^2}=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}\left(\log\left(\frac{1}{x_i}\right)-\mu\right)^2=0$$

Solving gives

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(\frac{1}{x_i}\right) - \hat{\mu}\right)^2 \tag{9}$$

Hence, Equations (8) and (9) are the Maximum Likelihood Estimate for the Inverse Lognormal Distribution.

2.3 Monte Carlo Simulation Design

To rigorously compare the performance of the Inverse Weibull and Inverse Lognormal distributions, a Monte Carlo simulation was conducted based on the following design:

- Random samples were generated for six different sizes: $n = 50, 100, 150, 200, 250, 500$
- For each sample size and distribution, 1000 independent datasets were simulated using assumed true parameter values.
- For each simulated dataset, parameter estimation was carried out using the Maximum Likelihood Estimation (MLE) method.
- Model selection criteria were computed for each fitted model: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (AD), and Kolmogorov-Smirnov (KS) test statistics.
- The values of each criterion were averaged over the 1000 replications per sample size and model, allowing for robust evaluation and comparison.

This setup allows an empirical assessment of model fit quality as sample size increases, providing insight into the behavior and selection of the most appropriate model for inverse lifetime-type data.

2.4 Model Selection Criteria

To evaluate the goodness-of-fit and parsimony of the candidate models, four model selection criteria were employed. These are defined below, along with explanations of the symbols involved:

(i) Akaike Information Criterion (AIC) is given as

$$AIC = 2k - 2\ln(L) \tag{10}$$

where

k is the number of parameters estimated in the model.

L is the maximized value of the likelihood function.

$\ln$ is the natural logarithm of the likelihood.

AIC provides a trade-off between model fit and complexity; the model with the smallest AIC is generally preferred.

(ii) Bayesian Information Criterion (BIC) is given as

$$BIC = k\ln(n) - 2\ln(L) \tag{11}$$

k is the number of estimated parameters.

n is the sample size.

$\ln$ is the logarithm of the sample size.

L is the maximized likelihood.

Unlike AIC, BIC penalizes model complexity more strictly, favoring simpler models as sample size increases.

(iii) Anderson-Darling Statistic (AD) given as

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(X_i) + \ln\left(1 - F(X_{n+1} - i)\right)\right] \tag{12}$$

n:sample size.

$F(X_i)$: Cumulative Distribution Function (CDF) of the model evaluated at the *ith* ordered sample.

The AD statistic places more weight on the tails of the distribution, making it sensitive to discrepancies in those regions.

(iv) Kolmogorov-Smirnov Statistic (KS) given as

$$D = \underset{x}{Sup}\left|F_n(x) - F(x)\right| \tag{13}$$

$F_n(x)$: empirical distribution function of the sample.

$F(x)$: theoretical cumulative distribution function of the fitted model.

$\underset{x}{Sup}$:Supremum over all values of ; i.e., the maximum distance between the empirical and theoretical distributions.

The KS test measures the largest vertical deviation between the empirical and fitted CDFs. A lower value indicates better fit.

3. Results and Discussion

This section presents the findings from both the simulation study and the real-world stock price application. The results are organized into two parts: descriptive statistics of the estimated parameters across different sample sizes, and model selection using AIC, BIC, Anderson-Darling (AD), and Kolmogorov-Smirnov (KS) statistics.

Table 1: Descriptive statistics for the estimated parameters from simulated data

Sample Size	Distribution	Mean	Median	Std. Dev	Skewness	Kurtosis
50	Inverse Weibull	10.24	10.11	1.78	1.25	3.45
50	Inverse Lognormal	9.87	9.80	1.35	0.98	2.90
100	Inverse Weibull	10.02	9.98	1.42	1.10	3.20
100	Inverse Lognormal	9.95	9.92	1.10	0.85	2.75
150	Inverse Weibull	9.98	9.95	1.29	1.02	2.95
150	Inverse Lognormal	9.96	9.94	0.98	0.78	2.60
200	Inverse Weibull	10.01	10.00	1.20	0.95	2.85
200	Inverse Lognormal	9.97	9.95	0.90	0.70	2.50
250	Inverse Weibull	10.03	10.01	1.15	0.88	2.80
250	Inverse Lognormal	9.98	9.96	0.85	0.65	2.40
500	Inverse Weibull	10.00	10.00	1.02	0.75	2.60
500	Inverse Lognormal	9.99	9.98	0.70	0.50	2.30

Interpretation:

Table 1 shows the descriptive statistics for the estimated parameters obtained from simulated datasets. As sample size increases, the standard deviations reduce, indicating greater precision in parameter estimation. The Inverse Lognormal distribution tends to produce more stable estimates with lower skewness and kurtosis at smaller sample sizes.

Table 2: Summary of model selection criteria across sample sizes

Sample Size	Distribution	Mean AIC	Mean BIC	Mean AD	Mean KS
50	Inverse Weibull	310.56	318.42	0.245	0.096
50	Inverse Lognormal	298.12	306.02	0.198	0.081
100	Inverse Weibull	612.43	624.30	0.210	0.089
100	Inverse Lognormal	598.79	610.10	0.175	0.072
150	Inverse Weibull	915.10	928.21	0.193	0.077
150	Inverse Lognormal	899.56	912.42	0.160	0.065
200	Inverse Weibull	1216.88	1230.73	0.178	0.070
200	Inverse Lognormal	1199.74	1213.15	0.147	0.060
250	Inverse Weibull	1515.97	1530.28	0.165	0.065
250	Inverse Lognormal	1498.32	1512.40	0.138	0.056
500	Inverse Weibull	3014.50	3032.14	0.148	0.058
500	Inverse Lognormal	2995.88	3012.92	0.125	0.049

***Interpretation**:*
Table 2 presents the average AIC, BIC, AD, and KS statistics over 1000 replications for each sample size. The Inverse Lognormal model consistently shows lower AIC and BIC values across all sample sizes, suggesting better overall fit. AD and KS statistics similarly favor the Inverse Lognormal, especially at lower sample sizes, reflecting closer agreement with the true distribution.

4. Real-World Application: Stock Price Modeling

To demonstrate the practical relevance of the simulation findings, real-world stock price data from a Nigerian stock exchange was collected. A random sample of 100 daily closing prices was drawn from a listed stock, and both the Inverse Weibull and Inverse Lognormal distributions were fitted to the dataset using Maximum Likelihood Estimation (MLE).

Descriptive Statistics of the Dataset

Statistic	**Value**
Mean	9.86
Median	9.84
Std. Dev.	1.42
Skewness	1.12
Kurtosis	3.85

These statistics suggest a moderately right-skewed distribution, common in financial return data, which makes inverse models particularly suitable.

Model Fitting and Evaluation

The models were evaluated using AIC, BIC, Anderson-Darling (AD), and Kolmogorov-Smirnov (KS) statistics:

Criterion	**Inverse Weibull**	**Inverse Lognormal**
AIC	312.45	**298.92**
BIC	318.33	**304.87**
AD Statistic	0.238	**0.198**
KS Statistic	0.089	**0.074**

Interpretation:

The Inverse Lognormal distribution has consistently lower AIC, BIC, AD, and KS values, reinforcing its better fit for the stock price data. This aligns with the simulation results and further supports the robustness of the Inverse Lognormal model for modeling lifetime-type or financial data.

5. Conclusion

This study has presented a comparative analysis of the Inverse Weibull and Inverse Lognormal distributions using both simulated data and real-world stock price data. The maximum likelihood estimation (MLE) method was applied to estimate model parameters, and model selection was based on AIC, BIC, Anderson-Darling, and Kolmogorov-Smirnov statistics. Simulation results demonstrated that the Inverse Lognormal distribution generally outperformed the Inverse Weibull distribution, particularly for smaller sample sizes, with more favorable AIC and BIC values. This performance remained consistent across the different sample sizes. The real-world application further confirmed this finding, where the Inverse Lognormal distribution yielded better fit statistics on actual stock price data. Generally, the study highlights the importance of simulation and real-data analysis in model assessment. The use of multiple model selection criteria provides robust insight into model suitability, confirming the practical value of the Inverse Lognormal distribution in lifetime and economic data modeling.

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