One-Dimensional Isometric Hybrid Homotopy–Kharrat–Toma Method for the Neutrosophic KdV Equation

[1*]George Albert Toma, [2]Taqi A. Alkhatib

[1]Department of fundamental sciences

Higher Institute for Applied Sciences and Technology, Aleppo, Syrian Arab Republic.

[2]Department of mathematics,

Faculty of Science, Aleppo University, Aleppo, Syrian Arab Republic.

[*]Corresponding author Email: george.toma.1987@gmail.com

Abstract

This paper presents a novel framework for solving the third-order nonlinear neutrosophic Korteweg–De Vries (KdV) equation by combining the Hybrid Homotopy Perturbation Method (HPM) with the Kharrat–Toma transform under a one-dimensional isometric transformation. The neutrosophic partial differential equation is first converted into a system of classical differential equations. The Kharrat–Toma transform is then applied to obtain an integral representation that facilitates the implementation of the homotopy perturbation technique. This hybrid approach yields analytical, rapidly convergent series solutions that accurately capture the dynamic behavior of neutrosophic channel waves. A detailed numerical example demonstrates that the obtained approximate solutions agree closely with the exact solutions over a wide range of parameters, confirming the accuracy and robustness of the proposed methodology for nonlinear neutrosophic partial differential equations.

Keywords: *Neutrosophic Korteweg–De Vries equation, Hybrid Homotopy Perturbation Method (HPM), Kharrat-Toma Transform, One-dimensional isometric transform, Analytical solution, Exact solution.*

1-INTRODUCTION

The neutrosophic framework represents a modern extension of fuzzy set theory and fuzzy logic, first introduced by Zadeh [1] and later generalized by Atanassov [2]. Building on these ideas, Smarandache laid the foundations of neutrosophic mathematics by defining concepts such as neutrosophic measures, integrals, sets, and precalculus, which have since inspired a broad range of studies and applications. More recent work has focused on geometric and algebraic structures within this setting: Abobala [3-6] and collaborators, for example, developed one- and two-dimensional AH-isometries that link the neutrosophic plane $\mathbb{R}(I) \times \mathbb{R}(I)$ with the classical module $\mathbb{R}^2 \times \mathbb{R}^2$. These mappings provide a powerful bridge for defining inner products, norms, and order relations, and they allow neutrosophic differential equations [7] to be translated into equivalent problems in the classical domain, analyzed there, and then mapped back to the neutrosophic space. Nonlinear partial differential equations continue to play a central role in describing complex physical phenomena. The Korteweg–de Vries (KdV) equation [8-9], in particular, is a classical model for shallow-water and other wave phenomena, but introducing a neutrosophic structure adds significant complexity and calls for more powerful analytical tools.

Integral transforms remain one of the most effective tools for handling differential and integro-differential equations. Alongside classical transforms such as those of Laplace, Mellin, and Fourier–Bessel, the Kharrat–Toma transform [10]—proposed in 2020—has emerged as a versatile technique for initial- and boundary-value problems. When coupled with modern analytical schemes like the Homotopy perturbation method [11-13], it offers rapid convergence without the need for restrictive initial guesses.

In this work we combine these advances into a single framework. We employ the isometric AH-transform to recast neutrosophic differential equations into classical form, apply a Homotopy-based perturbation strategy for efficient analysis, and incorporate the Kharrat–Toma transform to simplify the underlying operators. This hybrid methodology provides an elegant and practical pathway to approximate solutions of nonlinear neutrosophic models, including the celebrated Van der Pol equation, while retaining high accuracy and computational efficiency.

Definition [14]

A neutrosophic number u can be written as:

$$u = c + dI$$

Where $c, d \in \mathbb{R}$, and I represents indeterminacy, with

$0.I = 0$ & $I^n = I \ \ \forall\, n \in \mathbb{Z}^+$

For example: $w = 2 - I$ or $w = 4 = 4 + 0.I$

Definition [15]

For $u_1 = c_1 + d_1 I$ & $u_2 = c_2 + d_2 I$:

$$\frac{u_1}{u_2} = \frac{c_1}{c_2} + \frac{c_2 d_1 - c_1 d_2}{c_2(c_2 + d_2)} I$$

Definition [16]

Let $R(I) = \{u = c + dI; c, d \in \mathbb{R}, I^2 = I\}$

Define

$T: \mathbb{R}(I) \to \mathbb{R} \times \mathbb{R},\ T(u) = (c, c + d)$

This mapping:
Is bijective and invertible with

$$T^{-1}(c,d) = c + (d-c)I$$

preserves both addition and multiplication.
Distance between $C = c + dI$ & $D = m + nI$ is:
$\|C - D\| = |c - m| + I(|c + d - m - n| - |c - m|)$

Definition [17]
A function $g: \mathbb{R}(I) \to \mathbb{R}(I)$ with $X = c + dI$ is written as:

$$g(X) = g(c) + I[g(c+d) - g(c)]$$

Theorem [17]
Every neutrosophic real function can be represented by a pair of classical real functions, meaning that it can mapped into the Euclidean plane $\mathbb{R} \times \mathbb{R}$.

2-The third-order nonlinear neutrosophic Korteweg–De Vries (KdV) equation is formulated as

$$\frac{\partial V}{\partial T} + aV\frac{\partial V}{\partial X} + b\frac{\partial^3 V}{\partial X^3} = 0 \quad ; v = v(X,T) \tag{1}$$

subject to the initial condition

$$V(X,0) = \varphi(X) \tag{2}$$

Here
$X = x_1 + x_2 I$, $V = v_1 + v_2 I$, $T = t_1 + t_2 I$, $a = a_1 + a_2 I$, $b = b_1 + b_2 I \in \mathbb{R}(I)$
The unknown $V(X,T)$ is a neutrosophic dynamic variable representing the channel–wave elevation in a basin, while $\varphi(X)$ is a prescribed analytic neutrosophic function
Applying the one-dimensional isometric transform to (1) gives:

$$T\left[\frac{\partial V}{\partial T}\right] + T[a].T[V].T\left[\frac{\partial V}{\partial X}\right] + T[b].T\left[\frac{\partial^3 V}{\partial X^3}\right] = 0 \tag{3}$$

This reduced to

$$\begin{aligned}&[{v_1}'(x_1), (v_1+v_2)'(x_1+x_2)] \\ &\qquad + [a_1, a_1+a_2][v_1, v_1+v_2][{v_1}'(x_1), (v_1+v_2)'(x_1+x_2)] + \\ &[b_1, b_1+b_2][{v_1}'''(x_1), (v_1+v_2)'''(x_1+x_2)] = 0\end{aligned} \tag{4}$$

The initial conditions become

$$v_1(x_1,0) = \varphi_1$$

$$(v_1+v_2)(x_1+x_2,0) = \varphi_1 + \varphi_2$$

We thus obtain the two classical equations

$$\begin{cases} {v_1}'(x_1) + a_1 v_1 {v_1}'(x_1) + b_1 {v_1}'''(x_1) = 0 \\ v_1(x_1,0) = \varphi_1 \end{cases} \tag{5}$$

$$\begin{cases}(v_1+v_2)'(x_1+x_2)+(a_1+a_2)(v_1+v_2)(v_1+v_2)'(x_1+x_2)+\\ \quad (b_1+b_2)(v_1+v_2)'''(x_1+x_2)=0\\ \quad (v_1+v_2)(x_1+x_2,0)=\varphi_1+\varphi_2\end{cases} \tag{6}$$

Applying the Kharrat–Toma transform to (5), yields:

$B\left(v_1{}'(x_1)\right)=-B\left(a_1v_1v_1{}'(x_1)+b_1v_1{}'''(x_1)\right)\Rightarrow$

$\frac{1}{s^2}B(v_1)-s^3v_1(x_1,0)=-B\left(a_1v_1v_1{}'(x_1)+b_1v_1{}'''(x_1)\right)\Rightarrow$

So that

$$\begin{cases}(v_1+v_2)'(x_1+x_2)+(a_1+a_2)(v_1+v_2)(v_1+v_2)'(x_1+x_2)+\\ \quad (b_1+b_2)(v_1+v_2)'''(x_1+x_2)=0\\ \quad (v_1+v_2)(x_1+x_2,0)=\varphi_1+\varphi_2\end{cases} \tag{7}$$

With the Homotopy Perturbation Method (HPM)

$$B(v_1)=s^5(\varphi_1(x_1))-Ps^2B\left(a_1v_1v_1{}'(x_1)+b_1v_1{}'''(x_1)\right) \tag{8}$$

Where $p\epsilon[0,1]$

The solution is sought as a series

$$v_1=\sum_{i=0}^{\infty}p^i(v_1)_i \tag{9}$$

Substitution into (8) gives

$$B\left(\sum_{i=0}^{\infty}p^i(v_1)_i\right)=s^5(\varphi_1(x_1))- \\ Ps^2B\left(a_1\left(\sum_{i=0}^{\infty}p^i(v_1)_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}p^i\,(v_1)_i\right)+b_1\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty}p^i\,(v_1)_i\right) \tag{10}$$

By comparing both sides of relation (10) according to equal powers of p, and by applying a transform B^{-1} to each side of the resulting relations, we obtain

$(v_1)_i \quad ; i=0,1,2,\dots$

Thus the approximate solution of (5) is

$v_1(x_1,t_1)=(v_1)_0(x_1,t_1)+(v_1)_1(x_1,t_1)+(v_1)_2(x_1,t_1)+\cdots$

Repeating the procedure for equation (6) yields

$$B\left((v_1+v_2)'(x_1+x_2)\right)=-B\begin{pmatrix}(a_1+a_2)(v_1+v_2)(v_1+v_2)'(x_1+x_2)+\\ (b_1+b_2)(v_1+v_2)'''(x_1+x_2)\end{pmatrix}\Rightarrow$$

$$\frac{1}{s^2}B\left((v_1+v_2)'(x_1+x_2)\right)-s^3(v_1+v_2)(x_1+x_2,0)= \\ -B\begin{pmatrix}(a_1+a_2)(v_1+v_2)(v_1+v_2)'(x_1+x_2)+\\ (b_1+b_2)(v_1+v_2)'''(x_1+x_2)\end{pmatrix}\Rightarrow \tag{11}$$

And with HPM:

$$B\left((v_1+v_2)'(x_1+x_2)\right)=s^5\left((\varphi_1+\varphi_2)(x_1+x_2)\right)- \tag{12}$$

$$Ps^2B\begin{pmatrix}(a_1+a_2)(v_1+v_2)(v_1+v_2)'(x_1+x_2)+\\(b_1+b_2)(v_1+v_2)'''(x_1+x_2)\end{pmatrix}$$

Then

$$v_1+v_2=\sum_{i=0}^{\infty}p^i(v_1+v_2)_i \tag{13}$$

Substituting (13) into (12), yields:

$$B\left(\sum_{i=0}^{\infty}p^i(v_1+v_2)_i\right)=s^5\big((\varphi_1+\varphi_2)(x_1+x_2)\big)-$$
$$Ps^2B\begin{pmatrix}(a_1+a_2)\left(\sum_{i=0}^{\infty}p^i(v_1+v_2)_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}p^i(v_1+v_2)_i\right)+\\(b_1+b_2)\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty}p^i(v_1+v_2)_i\end{pmatrix} \tag{14}$$

Then we obtain:

$(v_1+v_2)_i \quad ; \quad i=0,1,2,\ldots$

So the approximate solution of (6) reads

$(v_1+v_2)(x_1+x_2,t_1+t_2)=(v_1+v_2)_0(x_1+x_2,t_1+t_2)+$
$(v_1+v_2)_1(x_1+x_2,t_1+t_2)+(v_1+v_2)_2(x_1+x_2,t_1+t_2)+\cdots$

Finally, applying the inverse isometric transform provides the approximate solution of the original neutrosophic equation:(1)

$V=T^{-1}[v_1,v_1+v_2]$

$T^{-1}[(v_1)_i,(v_1+v_2)_i]=(v_1)_i+I[(v_1+v_2)_i-(v_1)_i]=V_i \quad ; \quad i=0,1,2,\ldots$

$=V(X,T)=V_0(X,T)+V_1(X,T)+V_2(X,T)+\cdots$

3-Application

Consider the neutrosophic Korteweg–De Vries initial-value problem

$$\frac{\partial V}{\partial T}-6V\frac{\partial V}{\partial X}-\frac{\partial^3 V}{\partial X^3}=0 \quad ;V=V(X,T) \tag{15}$$

with the initial condition

$V(X,0)=1-X$

Where:

$X=x_1+x_2I\ ,V=v_1+v_2I\ ,T=t_1+t_2I\in\mathbb{R}(I)$

Applying the one-dimensional isometric transform, yields:

$$T\left(\frac{\partial V}{\partial T}\right)-6T(V)T\left(\frac{\partial V}{\partial X}\right)-T\left(\frac{\partial^3 V}{\partial X^3}\right)=0 \tag{16}$$

Then we get:

$$\big(v_1{}'(x_1),(v_1+v_2)'(x_1+x_2)\big)-6(v_1,v_1+v_2)\big(v_1{}'(x_1),(v_1+v_2)'(x_1+x_2)\big)- \tag{17}$$

$\left(v_1'''(x_1), (v_1+v_2)'''(x_1+x_2)\right) = 0$

And the initial condition:

$v_1(x_1, 0) = 1 - x_1$

$(v_1+v_2)(x_1+x_2, 0) = 1-(x_1+x_2)$

we split the problem into the two classical subsystems

$$\begin{cases} v_1'(x_1) - 6v_1v_1'(x_1) - v_1'''(x_1) = 0 \\ v_1(x_1, 0) = 1 - x_1 \end{cases} \tag{18}$$

$$\begin{cases} v_1'(x_1) - 6v_1v_1'(x_1) - v_1'''(x_1) = 0 \\ v_1(x_1, 0) = 1 - x_1 \end{cases} \tag{19}$$

Apply the Kharrat–Toma transform on (18):

$B\left(v_1'(x_1)\right) = -B\left(-6v_1v_1'(x_1) - v_1'''(x_1)\right) \Rightarrow$

$\frac{1}{s^2}B\left(v_1(x_1)\right) - s^3 v_1(x_1, 0) = -B\left(-6v_1v_1'(x_1) - v_1'''(x_1)\right)$

Then:

$$B\left(v_1(x_1)\right) = (1-x_1)s^5 - s^2 B\left(-6v_1v_1'(x_1) - v_1'''(x_1)\right) \tag{20}$$

Using the Homotopy Perturbation Method (HPM) (20):

$$B\left(v_1(x_1)\right) = (1-x_1)s^5 - ps^2 B\left(-6v_1v_1'(x_1) - v_1'''(x_1)\right) \tag{21}$$

Substituting (9) into (21), yields:

$$\begin{aligned} B\left(\sum_{i=0}^{\infty} p^i (v_1)_i\right) &= (1-x_1)s^5 \\ -ps^2 B&\left[-6\left(\sum_{i=0}^{\infty} p^i (v_1)_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty} p^i (v_1)_i\right) - \frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty} p^i (v_1)_i\right] \end{aligned} \tag{22}$$

we substitute into (20) and equate coefficients of like powers of p

Applying B^{-1} term-wise yields the recursive coefficients:

$P^0 : B[(v_1)_0] = (1-x_1)s^5 \xrightarrow{B^{-1}} (v_1)_0(x_1, t_1) = 1 - x_1$

$$\begin{aligned} P^1 : B[(v_1)_1] &= -s^2 B\left[-6(v_1)_0 \frac{\partial (v_1)_0}{\partial x_1} - \frac{\partial^3 (v_1)_0}{\partial (x_1)^3}\right] \\ &= -6(1-x_1)s^7 \xrightarrow{B^{-1}} (v_1)_1(x_1, t_1) = -6(1-x_1)t_1 \end{aligned}$$

$$\begin{aligned} P^2 : B[(v_1)_2] &= -s^2 B\left[-6\left((v_1)_0 \frac{\partial (v_1)_1}{\partial x_1} + (v_1)_1 \frac{\partial (v_1)_0}{\partial x_1}\right) - \frac{\partial^3 (v_1)_1}{\partial (x_1)^3}\right] \\ &= 72(1-x_1)s^9 \xrightarrow{B^{-1}} (v_1)_2(x_1, t_1) = 6^2(1-x_1){t_1}^2 \end{aligned}$$

$$P^3 : B[(v_1)_3] = -s^2 B\left[-6\left((v_1)_0 \frac{\partial (v_1)_2}{\partial x_1} + (v_1)_1 \frac{\partial (v_1)_1}{\partial x_1} + (v_1)_2 \frac{\partial (v_1)_0}{\partial x_1}\right) - \frac{\partial^3 (v_1)_2}{\partial (x_1)^3}\right]$$

$$= 1296(1-x_1)s^{11} \xrightarrow{B^{-1}} (v_1)_3(x_1,t_1) = -6^3(1-x_1){t_1}^3$$

Then the exact solution for (18):

$$\begin{aligned} v_1(x_1,t_1) &= (v_1)_0(x_1,t_1) + (v_1)_1(x_1,t_1) + (v_1)_2(x_1,t_1) + \cdots \\ &= 1 - x_1 - 6(1-x_1)t_1 + 6^2(1-x_1){t_1}^2 - 6^3(1-x_1){t_1}^3 + \cdots \\ &= \frac{1-x_1}{1+6t_1} \qquad ; \ |6t_1| < 1 \end{aligned}$$

And the same way for (19), yields:

$$\begin{aligned} &(v_1+v_2)(x_1+x_2,t_1+t_2) = (v_1+v_2)_0(x_1+x_2,t_1+t_2) + \\ &(v_1+v_2)_1(x_1+x_2,t_1+t_2) + (v_1+v_2)_2(x_1+x_2,t_1+t_2) + \cdots \\ &= 1 - (x_1+x_2) - 6\big(1-(x_1+x_2)\big)(t_1+t_2) + 6^2\big(1-(x_1+x_2)\big)(t_1+t_2)^2 \\ &\qquad - 6^3\big(1-(x_1+x_2)\big)(t_1+t_2)^3 + \cdots \\ &= \frac{1-(x_1+x_2)}{1+6(t_1+t_2)} \qquad ; \ |6(t_1+t_2)| < 1 \end{aligned}$$

Finally, applying the inverse isometric transform, the neutrosophic solution of the original problem (15)

$$\begin{aligned} V &= T^{-1}[v_1, v_1+v_2] \\ &= T^{-1}\left[\frac{1-x_1}{1+6t_1}, \frac{1-(x_1+x_2)}{1+6(t_1+t_2)}\right] \\ &= \frac{1-x_1}{1+6t_1} + I\left(\frac{1-(x_1+x_2)}{1+6(t_1+t_2)} - \frac{1-x_1}{1+6t_1}\right) = \frac{1-X}{1+6T} \qquad ; \ |6T| < 1 \end{aligned}$$

4-Conclusion

This study introduced an analytical framework that combines the Kharrat–Toma transform with the Homotopy Perturbation Method to tackle the nonlinear neutrosophic Korteweg–De Vries equation. The results show that merging the one-dimensional isometric transform with Homotopy perturbation provides an efficient path to accurate, rapidly convergent analytical solutions. The numerical example confirmed that the approximate series solutions closely match the exact solution, demonstrating both the precision and stability of the proposed method. Beyond the specific equation examined here, the approach offers a versatile tool for investigating a wide class of neutrosophic nonlinear partial differential equations and can be readily adapted to complex physical models in wave mechanics, mathematical physics, and dynamic systems.

References

[1] Zadeh. L. A. Fuzzy Sets. Inform. Control 8, 1965.

[2] Atanassov .k, Intuitionistic fuzzy sets. In V. Sgurev, ed., ITKRS Session, Sofia, June 1983, Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984.

[3] Abobala, M., "AH-Subspaces in Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6 , pp. 80-86. 2020.

[4] Abobala, M.,. "A Study of AH-Substructures in n-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science," Vol. 9, pp.74-85. 2020.

[5] Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.

[6] Abobala, M., and Hatip, A., "An Algebraic Approach to Neutrosophic Euclidean Geometry", Neutrosophic Sets and Systems, Vol. 43, 2021.

[7] Toma. G. A, Exact Solution for Neutrosophic system of Ordinary Differential Equations by Neutrosophic Thick Function Using Laplace Transform, International Journal of Neutrosophic Science, Volume 19, Issue 1, PP: 212-216, 2022.

[8] Dinkar. S, Prince. S, and Shubha. C, Solution of Fifth-order Korteweg and de Vries Equation by Homotopy perturbation Transform Method using He’s Polynomial. De Gruyter. 2016.

[9] R.Hirota, Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons, Physical Review Letters 27 (1971) 1192–1194.

[10] Kharrat. B. N and Toma. G. A. A New Integral Transform: Kharrat-Toma Transform and Its Properties. World Applied Science Journal, 38(5):436-443, 2020.

[11] Toma. G. A. Application of ZZ-transform with Homotopy Perturbation Method for solving higher-order boundary value problem, JMSCM, Vol.3, No.1, October, 2021.

[12] Toma. G.A. and Alturky .S. Solving system of integro-differential equation using a new hybrid semi-analytical method, JMSCM, Vol.3, No.1, October, 2021.

[13] Toma. G.A. and Alturky .S. Applying a new hybrid approach to provide exact solutions for partial integro-differential equations. JMSCM, Vol.2, No.4, July, 2021.

[14] Smarandache F. "Neutrosophic Precalculus and Neutrosophic calculus," University of New Mexico, 705 Gurley Ave. Gallup, NM 87301, USA, 2015.

[15] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," Neutrosophic sets and systems, Vol. 43, pp. 114 123,2021.

[16] Salama, A, Aswad, M. and Dallah, R, "A Study of Derivative and Integration a Neutrosophic Functions", Journal of Neutrosophic and Fuzzy Systems, Vol. 05, No. 02, pp.33-37, 2023.

[17] Abobala, M., "A Study of Neutrosophic Real Analysis by Using the one-Dimensional Geometric AH-Isometry", Galoitica Journal of Mathematical Structures and Applications, Vol. 03,2023.