Higher-Order Linear Neutrosophic ODEs via the One-Dimensional Isometric Transform

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Abstract

In this paper, a general class of inhomogeneous linear neutrosophic ordinary differential equations with constant coefficients and arbitrary order is investigated. By employing the one-dimensional isometric transform (Khatib–Abu Balla transform), the original neutrosophic problem is rigorously reduced to an equivalent system of classical linear ordinary differential equations.

The solution procedure begins with the analysis of the corresponding homogeneous equation, followed by the construction of particular solutions for the inhomogeneous neutrosophic equations, where the forcing terms may take exponential, trigonometric, or polynomial forms. The differential operator method is then applied within the classical framework, and the inverse isometric transform is used to recover the exact neutrosophic solutions.

The general solution of the inhomogeneous neutrosophic equation is obtained as the superposition of the homogeneous and particular solutions. Several illustrative examples are provided to demonstrate the effectiveness, consistency, and applicability of the proposed method in solving higher-order inhomogeneous neutrosophic differential equations.

Keywords*: Ordinary differential equation, Isometric transform, Differential operator method.*

1- INTRODUCTION

Recently, considerable attention has been directed toward neutrosophic mathematics, a novel branch of mathematics introduced by Florentin Smarandache. Neutrosophic logic, neutrosophic, neutrosophic sets, and neutrosophic probability are among its key innovations, all characterized by incorporating indeterminacy as an intrinsic component. A remarkable feature of neutrosophic logic is that it generalizes fuzzy logic while encompassing classical logic as a special case [1]. In 2015, Smarandache extended the theory by defining the concept of the continuation of a neutrosophic function [1]. Earlier, in 2014, he introduced the notions of neutro-oscillator differentials and mereo-derivatives [2], and in 2013 he developed neutrosophic integration and mereo-integrals, complementing classical definitions of limit, continuity, derivative, and integral [3].

Differential equations remain one of the most important branches of mathematics due to their extensive applications in science and engineering. In recent years, particular attention has been given to the thickness function and its role in neutrosophic differential equations. Researchers have applied it to derive solutions for systems of neutrosophic ordinary differential equations using the Laplace transform [4], and for nonlinear neutrosophic Van der Pol oscillatory problems [5]. Moreover, applications of neutrosophic concepts have expanded across multiple fields, including medical sciences [6–8], databases [9], and physics through neutrosophic triplet groups [9]. In addition, significant contributions have been made in neutrosophic topology [10–13] and analysis [14], while neutrosophic integration presents new opportunities in calculus, particularly for computing areas between two neutrosophic functions.

In this research, we introduce an alternative approach to neutrosophic differential equations using the isometric transform. This transform establishes a direct connection between classical and neutrosophic logic and, together with its inverse, provides a versatile analytical tool to investigate and solve higher-order neutrosophic differential equations. This approach aims to broaden the methodologies available for tackling such equations and to offer exact solutions within the neutrosophic framework.

In recent years, neutrosophic theory has demonstrated its effectiveness in various applied fields such as engineering, artificial intelligence, medical decision-making, and optimization problems [15–19].

2- PRELIMINARIE

Definition 2.1. Neutrosophic Real Number: [1]

Suppose that w is a neutrosophic number. Then, it can be expressed in the following standard form

$$w = a + bI$$

where a, b are real coefficients, and I represents the indeterminacy, with the properties

$0.I = 0$ and $I^n = I$ for all positive integers n.

For example: $w = 2 - I$ or $w = 4 = 4 + 0.I$

Definition 2.2. Division of neutrosophic real numbers: [1]

Suppose that w_1 and w_2 are two neutrosophic numbers, where:

$w_1 = a_1 + b_1 I$, $w_2 = a_2 + b_2 I$

Then, their quotient is given by

$$\frac{w_1}{w_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I$$

Definition 2.3. [19]

Let $\mathbb{R}(I) = \{a + bI; a, b \in \mathbb{R}\}$ where $I^2 = I$,be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows:

$$T: \mathbb{R}(I) \to \mathbb{R} \times \mathbb{R}$$

$$T(a + bI) = (a, a + b)$$

Remark 2.4. [19]

T is an algebraic isomorphism between two rings and has the following properties

T is bijective.

T preserves of addition and multiplication

$$T[(a + bI) + (c + dI)] = T(a + bI) + T(c + dI)$$

$$T[(a + bI).(c + dI)] = T(a + bI).T(c + dI)$$

Since T is bijective, it is invertible, with the inverse given by:

$$T^{-1}: R \times R \to R(I)$$

$$T^{-1}(a, b) = a + (b - a)I$$

T preserves of distances

The distance on $R(I)$ can be defined as follows:

Let $A = a + bI$, $B = c + dI$ be two neutrosophic real numbers, then:

$L = \|\overrightarrow{AB}\| = d[(a + bI, c + dI)] = |a + bI - (c + dI)| = |(a - c) + I(b - d)|$

$= |a - c| + I[|a + b - c - d| - |a - c|]$

On the other hand

$T(\|\overrightarrow{AB}\|) = (|a - c|, |(a + b) - (c + d)|) = \big(d(a, c), d(a + b, c + d)\big)$

$= d[(a, a + b), (c, c + d)] = d[T(a + bI), T(c + dI)] = \|T(\overrightarrow{AB})\|.$

This implies that the distance is preserved up to isometry. i.e.

$\|T(AB)\| = T(\|AB\|).$

Example

Let the real neutrosophic function be defined as.

$f: \mathcal{R}(I) \to \mathcal{R}(I)$

$f(X) = 2IX - 1 + I \qquad ; \qquad X = x + yI$

We can directly calculate $f(1 + I)$ as follows

$f(1 + I) = 2I(1 + I) - 1 + I = 2I + 2I^2 - 1 + I$

$\qquad = 2I + 2I - 1 + I = -1 + 5I$

Then

$$T\big(f(X)\big) = T(2I)T(X) + T(-1+I)$$
$$= (0,2)(x, x+y) + (-1,0)$$
$$= (-1, 2x+2y)$$
$$T\big(f(1+I)\big) = (-1,4) \implies f(1+I) = T^{-1}(-1,4) = -1+5I$$

Definition 2.5: [20]

Let $f: R(I) \to R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function of a single neutrosophic variable. It can be written as

$$f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$$

Definition 2.6:

Let $f: \mathbb{R}(I) \times \mathbb{R}(I) \to \mathbb{R}(I) \times \mathbb{R}(I)$; $f = f(X,Y)$ and $X = x_1 + x_2 I$, $Y = y_1 + y_2 I \in \mathbb{R}(I)$

$f(X,Y)$ can written as follows:

$$f(X,Y) = f(x_1, y_1) + I[f(x_1 + x_2, y_1 + y_2) - f(x_1, y_1)]$$

Theorem 2.7. [21]

Any neutrosophic real function can be represented by two classical real functions, i.e., mapped into the classical Euclidean plane $\mathbb{R} \times \mathbb{R}$.

Definition 2.8:

We say that the function $f(X)$, where $X = x + yI$, defined on $D \subseteq \mathbb{R}(I)$, is differentiable at the point $A = a_1 + a_2 I \in D$ is the following condition is satisfied.

$$\lim_{H \to 0+0I} \frac{f(X+H) - f(X)}{H} = B \quad ; \quad H = h_1 + h_2 I \ , \ B = b_1 + b_2 I$$

Definition 2.9:

Let $\boldsymbol{f}(\boldsymbol{X}, \boldsymbol{Y})$ be a function defined on $\boldsymbol{D} \subseteq \mathbb{R}(I) \times \mathbb{R}(I)$, where $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$ belong to $\mathbb{R}(I)$.

Then, the partial derivatives are defined as follows.

$$\frac{\partial f}{\partial X} = f_X = \lim_{\Delta X \to 0+0I} \frac{f(X + \Delta X, Y) - f(X,Y)}{\Delta X}$$

$$= \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, y_1) - f(x_1, y_1)}{\Delta x_1}$$
$$+ I \begin{bmatrix} \displaystyle\lim_{\Delta(x_1+x_2) \to 0} \frac{f\big(x_1 + x_2 + \Delta(x_1+x_2), (y_1+y_2) - f(x_1+x_2, y_1+y_2)\big)}{\Delta(x_1+x_2)} - \\ \displaystyle\lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, y_1) - f(x_1, y_1)}{\Delta x_1} \end{bmatrix}$$

$$\frac{\partial f}{\partial Y} = f_Y = \lim_{\Delta Y \to 0+0I} \frac{f(X, Y + \Delta Y) - f(X,Y)}{\Delta Y}$$

$$= \lim_{\Delta y_1 \to 0} \frac{f(x_1, y_1 + \Delta y_1) - f(x_1, y_1)}{\Delta y_1} + I \left[\lim_{\Delta(y_1+y_2) \to 0} \frac{f\big(x_1 + x_2, y_1 + y_2 + \Delta(y_1 + y_2) - f(x_1 + x_2, y_1 + y_2)\big)}{\Delta(y_1 + y_2)} - \lim_{\Delta y_1 \to 0} \frac{f(x_1, y_1 + \Delta y_1) - f(x_1, y_1)}{\Delta y_1} \right]$$

Theorem 2.10. [21]

Let $\mathbb{R}(I)$ be the field of neutrosophic numbers, and let $X = x + yI$. Then

$e^X = e^{x+yI} = e^x + I[e^{x+y} - e^x]$

Proof: Suppose $A = e^X$. Then

$T(A) = T(e^X) = T(e^{x+yI}) = e^{T(x+yI)} = e^{(x,x+y)} = (e^x, e^{x+y})$

$\Longrightarrow \ A = T^{-1}(e^x, e^{x+y}) = e^x + I(e^{x+y} - e^x)$

Theorem 2.11. [21]

Let $\mathbb{R}(I)$ be the field of neutrosophic numbers, and let $X = x + yI$. Then

$$sin(a + bI) = sina + I[sin(a + b) - sina]$$
$$cos(a + bI) = cosa + I[cos(a + b) - cosa]$$

Definition 2.12:

Let $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$ be elements of $\mathbb{R}(I)$,where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Then, we define the non-homogeneous linear neutrosophic linear differential equation with constant coefficients of order n in the form

$$Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_n Y = f(X) \quad ; \quad Y = Y(X) \tag{1}$$

Where $a_1, \ldots, a_n \in \mathbb{R}$

Theorem 2.13

Equation (1) is equivalent to two classical non-homogeneous linear differential equations with constant coefficients.

Proof: by applying the isometric transformation T to equation (1), we obtain

$T\big[Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_n Y\big] = T[f(X)] \ \ \Longrightarrow$

$T\big[Y^{(n)}\big] + a_1 T\big[Y^{(n-1)}\big] + \cdots + a_n T[Y] = T[f(X)] \quad \Longrightarrow$

$$\big[{y_1}^{(n)}, (y_1 + y_2)^{(n)}\big] + a_1\big[{y_1}^{(n-1)}, (y_1 + y_2)^{(n-1)}\big] + \cdots + a_n[y_1, y_1 + y_2] = [f_1(x_1), f_2(x_1 + x_2)]$$

This is equivalent to

$$\begin{cases} {y_1}^{(n)} + a_1 {y_1}^{(n-1)} + \cdots + a_n y_1 = f_1(x_1) \\ (y_1 + y_2)^{(n)} + a_1 (y_1 + y_2)^{(n-1)} + \cdots + a_n (y_1 + y_2) = f_2(x_1 + x_2) \end{cases} \tag{2}$$

Definition 2.14

We call the equation

$$Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_n Y = 0 \tag{3}$$

the corresponding homogeneous neutrosophic linear differential equation of equation (1).

Moreover, by applying the isometric transformation T to equation (3), we obtain two corresponding classical homogeneous linear differential equations for the two equations in (2) of the for

$$\begin{cases} y_1^{(n)} + a_1 y_1^{(n-1)} + \cdots + a_n y_1 = 0 \\ (y_1 + y_2)^{(n)} + a_1 (y_1 + y_2)^{(n-1)} + \cdots + a_n (y_1 + y_2) = 0 \end{cases}$$

Remark

To find the general solution of equation (1), we follow the differential operator method. Then we obtain an equation of the form

$$(D^n + a_1 D^{n-1} + \cdots + a_n) Y = f(X)$$

Let us denote

$$F(D) = D^n + a_1 D^{n-1} + \cdots + a_n$$

Since the isometric transform T is an algebraic isomorphism between the neutrosophic field and the classical Euclidean plane, its inverse T^{-1} guarantees that any solution of the decomposed classical system corresponds uniquely to a solution of the original neutrosophic equation. Therefore, the recombination of the classical solutions using T^{-1} is mathematically justified.

3- GENERAL SOLUTION OF THE HOMOGENEOUS EQUATION

We call the equation

$$F(D) = D^n + a_1 D^{n-1} + \cdots + a_n = 0$$

the characteristic equation corresponding to the homogeneous neutrosophic differential equation. In order to find the roots of this characteristic equation, let us set $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$ and distinguish the following cases:

Case 1:

If the characteristic equation has distinct real roots, then the general solution of the homogeneous equation is:

$$Y(X) = \alpha_1 e^{D_1 X} + \cdots + \alpha_n e^{D_n X} \quad ; \quad \alpha_1, \dots, \alpha_n \in \mathbb{R}$$

Case 2:

If the characteristic equation has repeated real roots repeated k times, say $D = \mu$ a root repeated k times, then the general solution of the homogeneous equation is:

$$Y(X) = (\alpha_1 + \alpha_2 X + \cdots + \alpha_k X^{k-1}) e^{\mu X} \quad ; \quad \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

Case 3:

Since the coefficients of the equation are real, the characteristic equation has a pair of complex conjugate roots, namely

$D_{1,2} = a \pm ib$ then the general solution of the homogeneous equation takes the form:

$$Y(X) = e^{aX}[a \cos(bX) + a\sin(bX)] \quad ; \quad a, b \in \mathbb{R}$$

Case 4:
If the characteristic equation has repeated complex conjugate roots $D_{1,2} = a \pm ib$ repeated k times, then the general solution of the homogeneous equation is

$$Y(X) = e^{aX}\begin{pmatrix}(\alpha_1 + \alpha_2 X + \cdots + \alpha_k X^{k-1})\cos(bX) + \\ (\beta_1 + \beta_2 X + \cdots + \beta_k X^{k-1})\sin(bX)\end{pmatrix}$$

Where $a, b, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{R}$

Remark: All previous cases are proved by applying the transformation T, which yields two classical solutions; then, by taking the inverse transformation, we obtain the desired neutrosophic solution.

4- FINDING THE PARTICULAR SOLUTION OF THE NON-HOMOGENEOUS EQUATION

Case 1: Exponential function

$$f(X) = Ae^{aX} \quad ; \quad A, a \in \mathbb{R}$$

We know that:

$$D\ e^{aX} = a\ e^{aX}$$
$$D^2 e^{aX} = a^2 e^{aX}$$
$$D^n e^{aX} = a^n e^{aX}$$

Applying the operator $\frac{1}{a^n D^n}$ to both sides of the last relation, we obtain

$$\frac{1}{a^n}\frac{D^n}{D^n}\ e^{aX} = \frac{1}{D^n}\ e^{aX}$$

Then

$$\frac{1}{D^n}\ e^{aX} = \frac{1}{a^n} e^{aX}\ , F(a) \neq 0$$

Equation (1) becomes

$$F(D)Y = f(X) \quad \Rightarrow \quad F(D)Y = Ae^{aX} \quad \Rightarrow \quad Y_p = \frac{Ae^{aX}}{F(D)}$$

Subcases:

$F(a) \neq 0$, then $Y_p = \frac{Ae^{aX}}{F(a)}$

If $F(a) = 0$ repeated k times, then:

$$F(D) = (D - a)^k f(D) \quad \Rightarrow \quad Y_p = \frac{1}{f(a)}\frac{X^k}{k!}Ae^{aX} \quad ; \quad f(a) \neq 0$$

Case 2: Polynomial function

$$f(X) = P_n(X) \quad \Rightarrow \quad Y_p = \frac{P_n(X)}{F(D)}$$

Here, the particular solution is obtained using geometric series expansions:
$$\frac{1}{1-Y} = 1 + Y + Y^2 + \cdots \quad ; |Y| < 1$$
$$\frac{1}{1+Y} = 1 - Y + Y^2 - \cdots \quad ; |Y| < 1$$

Case 3: Trigonometric function
$$f(X) = Acos(aX) + Bsin(aX)$$
Subcases:
If ia is a root of $F(D) = 0$, replace $D^2 = -a^2$, and if D appears in the denominator, multiply by the conjugate to simplify
$$Y_p = \frac{A}{F(-a^2)}\cos(aX) + \frac{B}{F(-a^2)}\sin(aX)$$
If ia is not a root of $F(D) = 0$ reduce to the exponential case using Euler's formula
$$e^{iaX} = \cos aX + i \sin aX$$
And
$$\cos aX = Re\left(e^{iaX}\right) \quad , \quad \sin aX = Im\left(e^{iaX}\right)$$

Case 4: Product of exponential with polynomial or trigonometric function
$$f(X) = Ae^{aX}g(X)$$
Where $g(X)$ is either a trigonometric function or a polynomial. Then
$$Y_p = \frac{1}{F(D)}Ae^{aX}g(X) = Ae^{aX}\frac{1}{F(D+a)}g(X)$$

5- APPLICATIONS

Example 1.
Consider the equation as follows:
$$\mathrm{Y}''' - 6\mathrm{Y}'' + 8\mathrm{Y}' = \mathrm{e}^{\mathrm{x}}$$
Applying the isometric transformations T with $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$,we get
$T[\mathrm{Y}'''] - 6T[\mathrm{Y}''] + 8T[Y] = T[\mathrm{e}^{\mathrm{x}}] \quad \Longrightarrow$
$[y_1''', (y_1 + y_2)'''] - 6[y_1'', (y_1 + y_2)''] + 8[y_1, y_1 + y_2] = [e^{x_1}, e^{x_1+x_2}]$
This gives two classical equations:
$$\begin{cases} y_1''' - 6y_1'' + 8y_1 = e^{x_1} \\ (y_1 + y_2)''' - 6(y_1 + y_2)'' + 8(y_1 + y_2) = e^{x_1+x_2} \end{cases}$$
Applying the differential operator, yields:
$$\begin{cases} (D^3 - 6D^2 + 8D)y_1 = e^{x_1} \\ (D^3 - 6D^2 + 8D)(y_1 + y_2) = e^{x_1+x_2} \end{cases}$$
Characteristic equation of homogeneous part:
$D^3 - 6D^2 + 8D = 0 \quad \Longrightarrow \quad D(D^2 - 6D + 8) = 0 \quad \Longrightarrow$
$D_1 = 0 \quad , \quad D_2 = 2 \quad , \quad D_3 = 4$
General solution of homogeneous equation:

$$\begin{cases} y_{c_1} = a_1 + b_1 e^{2x_1} + c_1 e^{4x_1} \\ y_{c_1} + y_{c_2} = a_2 + b_2 e^{2(x_1+x_2)} + c_2 e^{4(x_1+x_2)} \end{cases}$$

And Particular solution:

$$\begin{cases} y_{p_1} = \dfrac{1}{D(D-2)(D-4)} e^{x_1} = \dfrac{1}{3} e^{x_1} \\ y_{p_1} + y_{p_2} = \dfrac{1}{D(D-2)(D-4)} e^{x_1+x_2} = \dfrac{1}{3} e^{x_1+x_2} \end{cases}$$

Then

$$\begin{cases} y_1 = y_{c_1} + y_{p_1} = a_1 + b_1 e^{2x_1} + c_1 e^{4x_1} + \dfrac{1}{3} e^{x_1} \\ y_1 + y_2 = \left(y_{c_1} + y_{c_2}\right) + \left(y_{p_1} + y_{p_2}\right) = a_2 + b_2 e^{2(x_1+x_2)} + c_2 e^{4(x_1+x_2)} + \dfrac{1}{3} e^{x_1+x_2} \end{cases}$$

Hence, the general solution of equation after applying the inverse transformation T^{-1} is: $Y = y_1 + y_2 I = T^{-1}[y_1, y_1 + y_2]$

$$= T^{-1}\begin{bmatrix} a_1 + b_1 e^{2x_1} + c_1 e^{4x_1} + \dfrac{1}{3} e^{x_1}, \\ a_2 + b_2 e^{2(x_1+x_2)} + c_2 e^{4(x_1+x_2)} + \dfrac{1}{3} e^{x_1+x_2} \end{bmatrix}$$

$$= a_1 + b_1 e^{2x_1} + c_1 e^{4x_1} + \frac{1}{3} e^{x_1} + I \begin{bmatrix} a_2 + b_2 e^{2(x_1+x_2)} + c_2 e^{4(x_1+x_2)} + \dfrac{1}{3} e^{x_1+x_2} - \\ \left(a_1 + b_1 e^{2x_1} + c_1 e^{4x_1} + \dfrac{1}{3} e^{x_1}\right) \end{bmatrix}$$

$$= \left(a_1 + I[a_2 - a_1]\right) + \left(b_1 e^{2x_1} + I\left[b_2 e^{2(x_1+x_2)} - b_1 e^{2x_1}\right]\right) +$$
$$\left(c_1 e^{4x_1} + I\left[c_2 e^{4(x_1+x_2)} - c_1 e^{4x_1}\right]\right) + \left(\frac{1}{3} e^{x_1} + I\left[\frac{1}{3} e^{x_1+x_2} - \frac{1}{3} e^{x_1}\right]\right)$$

Then:

$$Y = A + Be^{2X} + Ce^{4X} + \frac{1}{3} e^{X}$$

Where $A = a_1 + Ia_2, B = b_1 + Ib_2, C = c_1 + Ic_2 \in \mathbb{R}(I)$

The transition from the neutrosophic solution expressed in terms of the variable X to its classical representation is achieved through the inverse isometric transform T^{-1}. Since the transform T is bijective and preserves algebraic operations, applying T^{-1} to the classical solution pair uniquely reconstructs the neutrosophic solution. This guarantees that the obtained classical expressions are consistent representations of the original neutrosophic solution.

Example 2.

Consider the equation as follows:

$$Y'' + 4Y + 4 = \sin X$$

Applying the isometric transformations T with $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$,we get:

$T[Y'' + 4Y + 4] = T[\sin X] \quad \Longrightarrow$

$T[Y''] + 4T[Y] + 4 = T[\sin X] \quad \Longrightarrow$

$[y_1'', (y_1 + y_2)''] + 4[y_1, y_1 + y_2] + 4 = [\sin x_1, \sin(x_1 + x_2)]$

This gives two classical equations:

$$\begin{cases} y_1'' + 4y_1 + 4 = \sin x_1 \\ (y_1 + y_2)'' + 4(y_1 + y_2) + 4 = \sin(x_1 + x_2) \end{cases}$$

Applying the differential operator, yields:

$$\begin{cases} (D^2 + 4D + 4)y_1 = \sin x_1 \\ (D^2 + 4D + 4)(y_1 + y_2) = \sin(x_1 + x_2) \end{cases}$$

Characteristic equation of homogeneous part:

$$D^2 + 4D + 4 = 0 \quad \Longrightarrow \quad (D + 2)^2 = 0 \quad \Longrightarrow \quad D = -2$$

General solution of homogeneous equation:

$$\begin{cases} y_{c_1} = (a_1 + b_1 x)e^{-2x_1} \\ y_{c_1} + y_{c_2} = (a_2 + b_2 x)e^{-2(x_1 + x_2)} \end{cases}$$

And Particular solution:

$$\begin{cases} y_{p_1} = \dfrac{1}{D^2 + 4D + 4} \sin x_1 = \dfrac{1}{4D + 3} \sin x_1 \\ y_{p_1} + y_{p_2} = \dfrac{1}{D^2 + 4D + 4} \sin(x_1 + x_2) = \dfrac{1}{4D + 3} \sin(x_1 + x_2) \end{cases}$$

$$\begin{cases} = \dfrac{4D - 3}{16D^2 - 9} \sin x_1 = -\dfrac{1}{25}(4\cos x_1 - 3\sin x_1) \\ = \dfrac{4D - 3}{16D^2 - 9} \sin(x_1 + x_2) = -\dfrac{1}{25}(4\cos(x_1 + x_2) - 3\sin(x_1 + x_2)) \end{cases}$$

Then

$$y_1 = y_{c_1} + y_{p_1} = (a_1 + b_1 x)e^{-2x_1} - \frac{1}{25}(4\cos x_1 - 3\sin x_1)$$

$$y_1 + y_2 = \left(y_{c_1} + y_{c_2}\right) + \left(y_{p_1} + y_{p_2}\right) = (a_2 + b_2 x)e^{-2(x_1 + x_2)}$$

$$-\frac{1}{25}(4\cos(x_1 + x_2) - 3\sin(x_1 + x_2))$$

Hence, the general solution of equation after applying the inverse transformation T^{-1} is: $Y = y_1 + y_2 I = T^{-1}[y_1, y_1 + y_2]$

$$= T^{-1}\begin{bmatrix} (a_1 + b_1 x)e^{-2x_1} - \dfrac{1}{25}(4cosx_1 - 3\sin x_1), \\ (a_2 + b_2 x)e^{-2(x_1 + x_2)} - \dfrac{1}{25}(4\cos(x_1 + x_2) - 3\sin(x_1 + x_2)) \end{bmatrix}$$

$$= (a_1 + b_1 x)e^{-2x_1} - \frac{1}{25}(4\cos x_1 - 3\sin x_1)$$

$$+ I\begin{bmatrix} (a_2 + b_2 x)e^{-2(x_1 + x_2)} - \dfrac{1}{25}(4\cos(x_1 + x_2) - 3\sin(x_1 + x_2)) - \\ \left((a_1 + b_1 x)e^{-2x_1} - \dfrac{1}{25}(4\cos x_1 - 3\sin x_1)\right) \end{bmatrix}$$

$$= (a_1 + b_1 x)e^{-2x_1} + I\left[(a_2 + b_2 x)e^{-2(x_1 + x_2)} - (a_1 + b_1 x)e^{-2x_1}\right]$$

$$-\frac{1}{25}(4\cos x_1 - 3\sin x_1) +$$

$$I\left[-\frac{1}{25}(4\cos(x_1 + x_2) - 3\sin(x_1 + x_2)) + \frac{1}{25}(4\cos x_1 - 3\sin x_1)\right]$$

Then:

$Y = (A + BX)e^{-2X} - \frac{1}{25}(4\cos X - 3\sin X)$

Where $A = a_1 + Ia_2, B = b_1 + Ib_2 \in \mathbb{R}(I)$

6- CONCLUSION

In this study, exact general solutions for a class of higher-order inhomogeneous linear neutrosophic ordinary differential equations with constant coefficients have been derived using the one-dimensional isometric transform and the differential operator method. By decomposing the original neutrosophic equation into an equivalent classical system, the complexity inherent in neutrosophic calculus is significantly reduced while preserving the structural properties of the solution.

The proposed framework provides a clear and systematic approach for constructing both homogeneous and particular solutions of inhomogeneous neutrosophic differential equations, and the inverse transform guarantees that the obtained solutions are valid within the neutrosophic setting. The presented examples confirm the accuracy and robustness of the method.

These results not only enrich the theoretical foundations of neutrosophic differential equations but also open the door to future applications in science and engineering where uncertainty and indeterminacy play a central role. The developed methodology may serve as a basis for further extensions to nonlinear or variable-coefficient neutrosophic models.

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