SOLVING A KLEIN - GORDON EQUATION NON LINEAR AS A GENERALIZED PROBLEM OF MOMENTS

[12]María B. Pintarelli

[1]*Departamento de Matematica de la Facultad de Ciencias Exactas*
Universidad Nacional de La Plata, LaPlata -1900. Argentina

[2]*Departmento de CienciasBasicasde la Facultad de Ingenieria*
Universidad Nacional de La Plata -1900. Argentina

Corresponding author Email: mariabpintarelli@gmail.com

Abstract

We consider the problem of finding a function $w(x,t)$ that satisfy the equation $w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t)$ under Dirichlet boundary conditions or Neumann boundary conditions. We will see that an approximate solution can be found using the techniques of generalized inverse problem of moments and find dimensions for the error of the estimated solution.

Keywords*: generalized moment problem; integral equations; hiperbolic equation, Cauchy conditions.*

INTRODUCTION

We want to find $w(x,t)$ such that

$$w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t)$$

in $E = \{(x,t);\ a_1 \leq x \leq b_1\ ;\ a_2 < t \leq b_2\}$ under Dirichlet boundary conditions or Neumann boundary conditions using the problem generalized moments techniques.
Where $w(x,t) \,\epsilon\, L^2(E)$, $\Phi(x,t)$ and $\chi(x,t,w)$ know, and $\frac{\eth\chi}{\eth w} \geq 0$

There are numerous works in the literature that deal with the numerical solution of a Klein Gordon equation.
For example, in [1], motivated by the prediction of a minimal measurable length at Planck scale found in many candidate theories of quantum gravity, we examine the Klein-Gordon equation with a $\lambda\phi^4$ interaction and a symmetry-breaking term, in the presence of a generalized uncertainty principle associated with a minimal length.

In [2] the nonlinear Klein–Gordon equation's exact solutions are obtained through the application of an appropriate transformation based on He's semi-inverse approach. This equation considered a generalization of other famous models in applied science, such as Phi-4 equation, Duffing equation, Fisher–Kolmogorov model through population dynamics and Hodgkin–Huxley equation that characterizes the propagation of electrical signals via nervous system
In [3]the aim of this work is to establish novel traveling wave solutions of the nonlinear Atangana conformable Klein - Gordon equation using a new extended direct algebraic technique
In [4] the existence of multiple solutions for a class of Klein–Gordon equations coupled with Born–Infeld theory was investigated.
In [5] a hybrid numerical scheme based on Lucas and Fibonacci polynomials in combination with Stormer's method ¨ for the solution of Klein/Sinh-Gordon equations is proposed.
In [6] the main objective is to present an approximate numerical solution of the Klein – Gordon equation.
In [7] the exact analytical solutions of the 2-dimensional nonlinear Klein-Gordon equation (NLKGE) is investigated using the 3-dimensional Laplace transform method in conjunction with the Daftardar-Gejji and Jafari Method (Iterative method).
In [8] due to the importance of the generalized nonlinear Klein-Gordon equation (NL-KGE) in describing the behavior of light waves and nonlinear optical materials, including phenomena such as optical switching by manipulating the dispersion and nonlinearity of optical fibers and stable solitons, it is explained the approximation of this model by evaluating different classical and fractional terms.
[9] studies the Klein-Gordon equation coupled with the Born-Infeld (BI) theory with critical growth.
[10] presents a method of lines solution based on the reproducing kernel Hilbert space method to the nonlinear one-dimensional Klein-Gordon equation that arises in many scientific fields areas.
In [11] it is being investigated the small data global existence and pointwise decay of solutions to two systems of coupled wave-Klein– Gordon equations in two spatial dimensions.
In [12] is studied the existence of nontrivial solutions for nonlinear Klein– Gordon equations coupled with Born–Infeld theory with critical Sobolev exponents by variational methods.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation. The interest is not to compare with the existing methods, but to present a different method.

The generalized moments problem [13, 14, 15, 16, 17] is to find a function $f(x)$about a domain $\Omega \subset \mathrm{R}^{\mathrm{d}}$ that satisfies the sequence of equations

$$\mu_i = \int_\Omega g_i(x) f(x) dx \quad i \epsilon N ------------ (1)$$

where N is the set of the natural numbers, $(\mathrm{g_i(x)})$ is a given sequence of functions in $\mathrm{L}^2(\Omega)$ linearly independent known and the succession of real numbers μ_i are known data.
The problem of Hausdorff moments [15, 17], is to find a function $f(x)$en (a, b) such that

$$\mu_i = \int_a^b x^i\, f(x)dx \quad i\epsilon N - - - - - - - - -(2)$$

In this case $g_i(x) = x^i$ with i belonging to the set N.
If the integration interval is $(0, \infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty, \infty)$ we have the problem of Hamburger moments [15, 17].
The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [15, 16]. There are several methods to build regularized solutions. One of them is the truncated expansion method [16].
This method is to approximate (1) with the finite moments problem

$$\mu_i = \int_\Omega g_i(x) f(x)dx \quad i = 1,2, \dots, n - - - - - - - - - - (3)$$

where it is considered as approximate solution of $f(x)$ to

$$p_n(x) = \sum_{i=0}^{n} \lambda_i\, \phi_i(x)$$

and the functions $\phi_i(x)_{i=0,1,\dots,n}$ result of orthonormalize $\langle g_1, g_2, \dots, g_n \rangle$ being λ_i the coefficients based on the data μ_i . In the subspace generated by $\langle g_1, g_2, \dots, g_n \rangle$ the solution is stable. If $n\, \epsilon N$ is chosen in an appropriate way then the solution of (3) it approaches the solution of the original problem (1).
In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution ([16] p.p. 19 - 30).

ARTICLE ORGANIZATION

We want to find $w(x, t)$ such that

$$w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t)$$

under the Dirichlet or Neumann conditions using the problem generalized moments techniques
We will see that both problems can be solved in two steps.

First we find find $w(x, t)$ under Dirichlet conditions. Secondly, under Newmann conditions.
Finally the numerical examples and the conclusions.

APPROXIMATION OF $w(x, t)$ UNDER DIRICHLET CONDITIONS

We want to find $w(x, t)$ such that

$$w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t) - - - - - - -(4)$$

about a domain $E = (a_1, b_1) \times (a_2, b_2)$ with the conditions

$$w(x, a_2) = k_1(x) \quad a_1 \leq x \leq b_1 \qquad w_t(x, a_2) = g_1(x) \quad a_1 \leq x \leq b_1$$
$$w(x, b_2) = k_2(x) \quad a_1 \leq x \leq b_1 \qquad w_t(x, b_2) = g_2(x) \quad a_1 \leq x \leq b_1$$

$$w(b_1, t) = s_1(t) \quad w(a_1, t) = s_2(t) \quad a_2 < t \leq b_2$$

We considerer

$$w_{xx}(x,t) - w_{tt}(x,t) = \chi(x,t,w) - \Phi(x,t)$$

We take as an auxiliary function

$$u(m,r,x,t) = e^{-(m+1)(x)} e^{-(r+1)(t)}$$

We define the vector field

$$F^* = \big(F_1(w), F_2(w)\big) = (w_x\,, -w_t)$$

We have

$$div(F^*) = w_{xx}(x,t) - w_{tt}(x,t) = \chi(x,t,w) - \Phi(x,t) = G1(x,t) - - - - - (5)$$

Moreover, as $udiv(F^*) = div(uF^*) - F^*. \nabla u$, then

$$\iint_E u\, div(F^*) dA = \iint_E div(uF^*) dA - \iint_E F^* . \nabla u dA - - - - - - - - - - (6)$$

where $\nabla u = (u_x, u_t)$

Besides that

$$\iint_E div(uF^*) dA = \iint_E (uw_x)_x - (uw_t)_t dA =$$

$$= \iint_E u\, div(F^*) dA + \iint_E (u_x w_x - u_t w_t) dA - - - - - - - - - - - - - - - - (7)$$

Then from (6) y (7):

$$\iint_E (u_x w_x - u_t w_t) dA = \iint_E F^* . \nabla u dA - - - - - - - - - - - - - - (8)$$

On the other hand, it can be proven, after several calculations that, integrating by parts:

$$\iint_E (u_x w_x - u_t w_t) dA = \iint_E (w_x(-(m+1)u) dA +$$

$$+ \int_{a_1}^{b_1} (r+1)(w(x,b_2)u(m,r,x,b_2) - w(x,a_2)u(m,r,x,a_2))dx -$$

$$-\frac{(r+1)^2}{m+1} \int_{a_2}^{b_2} (w(b_1,t)u(m,r,b_1,t) - w(a_1,t)u(m,r,b_1,t))dt + \frac{(r+1)^2}{m+1} \iint_E (w_x u) dA$$

If $m = r$,

$$\iint_E u(w_x - w_t) dA = \frac{\varphi(r,r)}{r}$$

with

$$\frac{\varphi(r,r)}{r} =$$
$$= \int_{a_2}^{b_2} u(r,r,b_1,t)w(b_1,t) - u(r,r,a_1,t)w(a_1,t)dt +$$
$$- \int_{a_1}^{b_1} u(r,r,x,b_2)w(x,b_2) - u(r,r,x,a_2)w(x,a_2)dx$$

We note $\varphi_1(r) = \frac{\varphi(r,r)}{r}$, then

$$\iint_E u(w_x - w_t) dA = \varphi_1(r) - - - - - - - - - - - - - (10)$$

To solve this integral equation we take a base $\psi_i(r) = r^i e^{-r} \quad i = 0,1,2,\dots,n$ of $L^2(E)$.
Then we multiply both members of (10) by $\psi_i(r) = r^i e^{-r}$ and we integrate with respect to r, we obtain

$$\iint_E H_i(x,t)(w_x - w_t) dA = \int_{a_2}^{b_2} \varphi_1(r)\psi_i(r) dr = \mu_i \quad i = 0,1,2,\dots,n - - - - (11)$$

where $H_i(x,t) = \int_{a_2}^{b_2} u(r,r,x,t)\psi_i(r)dr$.
We can interpret (11) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation $p_{1n}(x,t)$ for $w_x - w_t$.

SOLUTION OF THE GENERALIZED MOMENTS PROBLEM

We can apply the detailed truncated expansion method in [17] and generalized in [19] and [20] to find an approximation $p_{1n}(x,t)$ of $w_x - w_t$ for the corresponding finite problem with $i = 0,1,2,\dots,n$, where n is the number of moments μ_i.
We consider the basis $\phi_i(x,t) \quad i = 0,1,2,\dots,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t) \quad i = 0,1,2,\dots,n$.
We approximate the solution $w_x - w_t$ with [13] and generalized in [19] y [20]:

$$p_{1n}(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{where} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\,\mu_j \qquad i = 0,1,2,\dots,n$$

and the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) | \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) . \|\phi_i(x,t)\|^{-1} \quad 1 < i \le n \,; 1 \le j < i \,.$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0,1,\dots,n.$$

The proof of the following theorem is in [19] and [20].
In [20] the demonstration is made for b_2 finite. En [18] the demonstration is made for the one-dimensional case.
This Theorem gives a measure about the accuracy of the approximation.

Theorem
Let $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2\big((a_1,b_1)\times(a_2,b_2)\big)$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n} \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} H_i(x,t) f(x,t) dx\, dt - \mu_i \right|^2 \le \varepsilon^2$$

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} ((b_1 - a_1)^2 f_x^2 + (b_2 - a_2)^2 f_t^2) dx\, dt \le M^2 ------------ (12)$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} |f(x,t)|^2\, dt dx \le min_i \left\{ \|C^T C\| \varepsilon^2 + \frac{M^2}{8(n+1)^2} ; i = 0,1,\dots,n \right\}$$

where C it is a triangular matrix with elements $C_{ij}, \quad 1 < i \leq n;\ 1 \leq j < i$
and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} |p_{1n}(x,t) - f(x,t)|^2\,dxdt \leq \|CC^T\|\varepsilon^2 + \frac{M^2}{8(n+1)^2} ------------ (13)$$

So we have an equation in first order partial derivatives of the form

$$w_x(x,t) - w_t(x,t) = p_{1n}(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t) + A_2(x,t)w_t(x,t) = p_{1n}(x,t)$$

where $A_1(x,t) = 1$ and $A_2(x,t) = -1$.
It is resolved as in [18], that is, we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)w(x,t)dtdx = \varphi_2(m,r) ------- (14)$$

With $K(m,r,x,t) = u(m,r,x,t)(m_1(m+1) - m_2(r+1))$ where now it is taken as an auxiliary function

$$u(m,r,x,t) = e^{-m_1(m+1)(x)}e^{-m_2(r+1)(t)}$$

We take $m_1 = 0.5;\ m_2 = 1$

and

$$\varphi_2(m,r) = \int_{a_1}^{b_1} u(m,r,x,b_2)w(x,b_2) - u(m,r,x,a_2)w(x,a_2)dx -$$
$$- \int_{a_2}^{b_2} u(m,r,b_1,t)\,w(b_1,t) - u(m,r,a_1,t)w(a_1,t)dt - \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_{1n}(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)} \qquad i = 0,1,\dots,n_1 \quad j = 0,1,2,\dots,n_2$$

and we multiply both members of (14) by $\psi_{ij}(m,r)$ and we integrate with respect to m and r

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(x,t)H_{ij}(x,t) = \mu_{ij} ----------- (15)$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi_2(m,r)\psi_{ij}(m,r)dmdr$$

$$H_{ij}(x,t) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)\psi_{ij}(m,r)dmdr$$

We apply the truncated expansion method and find a numerical approximation $p_{2n}(x,t)$ for $w(x,t)$.

APPROXIMATION OF $w(x,t)$ UNDER MEUMANN CONDITIONS

We want to find $w(x,t)$ such that

$$w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t) \;-------- (16)$$

about a domain $E = (a_1,b_1)\times(a_2,b_2)$ with the conditions

$$\begin{array}{ll} w(x,a_2) = k_1(x) \quad a_1 \le x \le b_1 & w_t(x,a_2) = g_1(x) \quad a_1 \le x \le b_1 \\ w(x,b_2) = k_2(x) \quad a_1 \le x \le b_1 & w_t(x,b_2) = g_2(x) \quad a_1 \le x \le b_1 \end{array}$$

$$w_x(b_1,t) = s_1(t) \quad w_x(a_1,t) = s_2(t) \quad a_2 < t \le b_2$$

We considerer

$$w_{xx}(x,t) + w_{tt}(x,t) = 2w_{xx}(x,t) - \chi(x,t,w) + \Phi(x,t) = G2(x,t)$$

We take as an auxiliary function

$$u(m,r,x,t) = Cos(m\pi x)\, e^{-(r)(t)}$$

We define the vector field

$$F^* = \big(F_1(w),F_2(w)\big) = (w_x\,,w_t)$$

We considerer

$$\iint_E u\, G2(x,t)dA = \iint_E div(uF^*)dA - \iint_E F^*.\nabla u dA = \oint (uF^*).n ds - \iint_E F^*.\nabla u dA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_E F^*.\nabla u dA = \iint_E (F_1 u_x + F_2 u_t)dA$$

Integrating by parts:

$$\iint_E F_1\, u_x dA = \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_1\, u_x dx dt =$$

$$= \int_{a_2}^{b_2}\big(-w(b_1,t)u_x(m,r,b_1,t) - (-w(a_1,t))u_x(m,r,a_1,t)\big)dt - \iint_E w u_{xx}\, dA$$

Analogously

$$\iint_E F_2\, u_t dA = \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_2\, u_t dx dt =$$

$$= \int_{a_1}^{b_1}\big(-w(x,b_2)u_t(m,r,x,b_2) + w(x,a_2)u_t(m,r,x,a_2)\big)dx - \iint_E w u_{tt}\, dA$$

then

$$\iint_D F^*.\nabla u dA =$$

$$= \int_{a_1}^{b_1}\big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx + \iint_E w(u_{xx} + u_{tt})dA$$

where

$$\iint_E w(u_{xx} + u_{tt})dA = \iint_E wu((m\pi)^2 - (r+1)^2)dA$$

On the other hand,

$$\iint_E div(uF^*)dA = \int_C (uF^*).nds =$$

$$= \int_{a_1}^{b_1} u(m,r,x,b_2)w_t(x,b_2)dx - \int_{a_1}^{b_1} u(m,r,x,a_2)w_t(x,b_2)dx +$$

$$+ \int_{a_2}^{b_2} u(m,r,b_1,t)(w_x(b_1,t))dt - \int_{a_2}^{b_2} u(m,r,a_1,t)(w_x(a_1,t))dt = G(m,r)$$

$$\therefore \iint uG2(x,t)dA = G(m,r) -$$

$$-\int_{a_1}^{b_1} \big(w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2)\big)dx + \iint_E wu\left((m\pi)^2 - (r)^2\right)dA$$

We make $(m\pi)^2 - (r)^2 = 0$, then

$$\iint u(m,m\pi,x,t)G2(x,t)dA = G(m,m\pi) -$$
$$-\int_{a_1}^{b_1} \big(w(x,b_2)u_t(m,m\pi,x,b_2) - w(x,a_2)u_t(m,m\pi,x,a_2)\big)dx$$
$$= \varphi_3(m) - - - - - - - - -(17)$$

We take a base:

$$\psi_i(m) = m^i e^{-m} \qquad i = 0,1,\dots,n$$

and we multiply both members of (17) by $\psi_i(m)$ and we integrate with respect to m.

We apply the truncated expansion method and find a numerical approximation $p_{1n}(x,t)$ for $G2(x,t)$.

So we have an equation partial derivatives of the form

$$w_{xx}(x,t) + w_{tt}(x,t) = p_{1n}(x,t)$$

We take as an auxiliary function

$$u(m,r,x,t) = Cos(\frac{m\pi x}{b_1})\, e^{-(r+1)(t)}$$

We define the vector field

$$F^* = \left(F_1(w), F_2(w)\right) = (w_x \, , w_t)$$

Again we considerer

$$\iint_E u\, G2(x,t) dA = \iint_E div(uF^*) dA - \iint_E F^* . \nabla u dA = \oint (uF^*). nds - \iint_E F^* . \nabla u dA$$

And

$$\iint_E F^* . \nabla u dA = \iint_E (F_1 u_x + F_2 u_t) dA$$

Integrating by parts:

$$\iint_E F_1\, u_x dA = \int_{a_2}^{b_2} \int_{a_1}^{b_1} F_1\, u_x dx dt =$$

$$= \int_{a_2}^{b_2} \left(w(b_1,t) u_x(m,r,b_1,t) - w(a_1,t) u_x(m,r,a_1,t)\right) dt - \iint_E w u_{xx}\, dA$$

If $m \epsilon N$

$$u_x(m,r,b_1,t) = -e^{-r(t+1)} sen\left(\frac{m\,\pi}{b_1} b_1\right) \frac{m\,\pi}{b_1} = 0$$

$$u_x(m,r,a_1,t) = -e^{-r(t+1)} sen(m\pi\, a_1) \frac{m\,\pi}{b_1} \underbrace{=}_{a_1=0} 0$$

Analogously

$$\iint_E F_2\, u_t dA = \int_{a_2}^{b_2} \int_{a_1}^{b_1} F_2\, u_t dx dt =$$

$$= \int_{a_1}^{b_1} \left(w(x,b_2) u_t(m,r,x,b_2) - w(x,a_2) u_t(m,r,x,a_2)\right) dx - \iint_E w u_{tt}\, dA$$

If $m \epsilon N$ and $a_1 = 0$ then

$$\iint_E F^* . \nabla u dA = \int_{a_1}^{b_1} \big(w(x, b_2) u_t(m, r, x, b_2) - w(x, a_2) u_t(m, r, x, a_2) \big) dx - \iint_E w(u_{xx} + u_{tt}) dA$$

Where

$$\iint_E w(u_{xx} + u_{tt}) dA = \iint_E wu \left(-\left(\frac{m\pi}{b_1}\right)^2 + (t+1)^2 \right) dA$$

On the other hand,

$$\int_C (uF^*) . nds =$$

$$= -\int_{a_1}^{b_1} u(m, r, x, a_2) w_t(x, a_2) dx + \int_{a_1}^{b_1} u(m, r, x, b_2) w_t(x, b_2) dx +$$

$$+ \int_{a_2}^{b_2} u(m, r, b_1, t) w_x(b_1, t) dt - \int_{a_2}^{b_2} u(m, r, a_1, t) w_x(a_1, t) dt = G(m, r)$$

Therefore

$$\iint uG2(x, t) dA =$$

$$G(m, r) - \int_{a_1}^{b_1} \big(w(x, b_2) u_t(m, r, x, b_2) - w(x, a_2) u_t(m, r, x, a_2) \big) \; dx + \iint_D wu \left(-\left(\frac{m\pi}{b_1}\right)^2 + (r+1)^2 \right) dA$$

Then

$$\iint_E wu \left(-\left(\frac{m\pi}{b_1}\right)^2 + (r+1)^2 \right) dA =$$

$$= -\int_{a_1}^{b_1} \big(w(x, b_2) u_t(m, r, x, b_2) - w(x, a_2) u_t(m, r, x, a_2) \big) dx - G(m, r) + \iint_E u \, p_{1n}(x, t) dA = \varphi(m, r)$$

Therefore

$$\iint_E wu\, dA = \frac{\varphi(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2 + (r+1)^2\right)}$$

To solve this integral equation, we give integer values to *m* and *r*:

$$m = 0,1,2, \ldots n_1 - 1 \quad ; \quad r = 1,2, \ldots, n_2$$

Then

$$\iint_E w(x,t) H_{mr}(x,t) dA = \frac{\varphi(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2 + (t+1)^2\right)} = \mu_{mr} \text{ ------------------ } (18)$$

We interpret (18) as a moments problem of two-dimensional generalized, $p_n(x,t)$ is the numerical approximation with the truncated expansion method for $w(x,t)$, with $n = n_1.n_2$

$$H_{mr}(x,t) = u(m,r,x,t) \quad m = 0,1,2, \ldots n_1 - 1 \quad ; \quad r = 1,2, \ldots, n_2$$

NUMERICALS EXAMPLES

Example 1

We consider the equation

$$w_{tt}(x,t) - w_{xx}(x,t) - \left(\tfrac{x^2}{4}\right) w(x,t) = \tfrac{1}{2} e^{-x-t}(2x-1) Cos\left(\tfrac{x^2}{4}\right) \qquad E = (0,2) \times (0,3)$$

Dirichlet conditions

The solution is : $w(x,t) = Sin\left(\frac{x^2}{4}\right) e^{-x-t}$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$
To find $p_{1n}(x,t)$ we take $n = 8$ moments
To find $p_{2n}(x,t)$ we take $n = 9$, moments $n_1 = 3, n_2 = 3$, in total $n = n_1.n_2 = 9$ moments.

$$\sqrt{\int_0^3 \int_0^2 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.0290442$$

In the Figure 1 we show the exact solution $w(x,t)$ (light blue) and the approximate solution $p_{2n}(x,t)$ (magenta color).

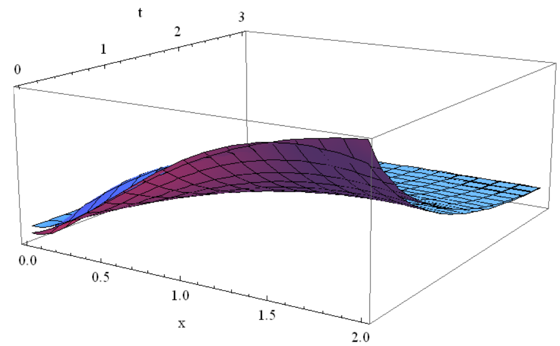

Fig. 1: $w(x,t)$ and $p_{2n}(x,t)$

Example 2

We consider the equation

$w_{tt}(x,t) - w_{xx}(x,t) - w(x,t)^3 = e^{-x-t} \qquad E = (0,1) \times (0,1)$

Neumann conditions

The solution is : $w(x,t) = e^{-\frac{x}{3}-\frac{t}{3}}$

Taking into account the previous theorem we calculate the accuracy as a way of comparing $w(x,t)$ with $p_{2n}(x,t)$

To find $p_{1n}(x,t)$ we take $n = 5$ moments

To find $p_{2n}(x,t)$ we take $n = 6$, moments $n_1 = 3, n_2 = 2$, in total $n = n_1.n_2 = 6$ moments.

$$\sqrt{\int_0^1 \int_0^1 \left(w(x,t) - p_{2n}(x,t)\right)^2 dxdt} = 0.00735342$$

In the Figure 2 we show the exact solution $w(x,t)$ (light blue) and the approximate solution $p_{2n}(x,t)$ (magenta color)

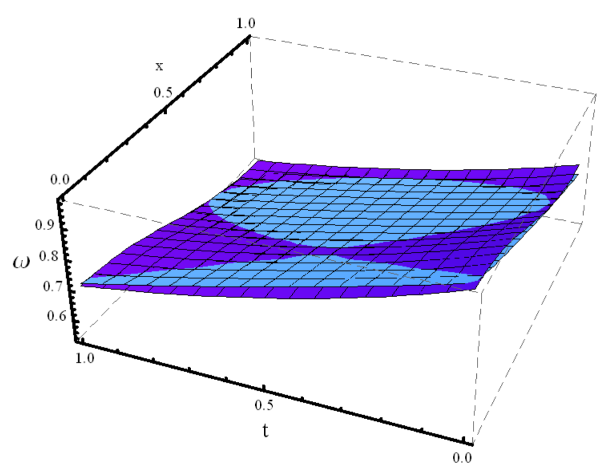

Fig. 2: $w(x,t)$ and $p_{2n}(x,t)$

CONCLUSIONS

The problem of finding $w(x,t)$ in the equation $w_{tt}(x,t) - w_{xx}(x,t) + \chi(x,t,w) = \Phi(x,t)$ about a domain $E = (a_1, b_1) \times (a_2, b_2)$, under the Dirichlet or Neumann conditions it is possible to solve it using the problem generalized moments techniques.

We can deduce $w(x,t)$ in two steps.

We consider the equation $w_{xx}(x,t) - w_{tt}(x,t) = G1(x,t)$ if we considerer Dirichlet conditions or $w_{xx}(x,t) + w_{tt}(x,t) = G2(x,t)$ if we considerer Neumann conditions (in this case $a_1 = 0$).

We turn it into an integral equation and with the techniques of inverse problem moments we find an approximate solution $p_{1n}(x,t)$ for $G1 \; or \; G2$.

Then we consider the equation $w_x(x,t) \mp w_t(x,t) = p_{1n}(x,t)$ and again we take it to an integral equation to solve it and find an approximate solution $p_{2n}(x,t)$ for $w(x,t)$.

REFERENCES:

1. Narges Heidari, Marc de Montigny, Ali Ahmadi Azar, Thambiayya Sathiyaraj, Hassan Hassanabadi; "Solutions of the nonlinear Klein-Gordon equation and the generalized uncertainty principle with the hybrid analytical and numerical method". Nucl. Phys. B 1009 ,116750, (2024)

2 Abdulhamed Alsisi; "Analytical and numerical solutions to the Klein–Gordon model with cubic nonlinearity". Alexandria Engineering Journal 99 ,31–37, (2024).

3. Hadi Rezazadeh , Mohammad Ali Hosseinzadeh, Lahib Ibrahim Zaidan , Fatima SD. Awad, Fiza Batool, Soheil Salahshour; "Innovative approache for the nonlinear atangana conformable Klein-Gordon equation unveiling traveling wave patterns". Partial Diǫerential Equations in Applied Mathematics 12 ,100935, (2024).

4. Jiayi Fei, Qiongfen Zhang; "On solutions for a class of Klein–Gordon equations coupled with Born–Infeld theory with Berestycki–Lions conditions on R ^{3}". Electronic ResearchArchive, , 32(4): 2363–2379, (2024).

5. Ihteram Ali, Imtiaz Ahmad; "Applications of the nonlinear Klein/Sinh-Gordon equations in modern physics: a numerical study". Mathematical Modelling and Control, 4(3): 361–373, (2024).

6. Francisco J. Cardona-García, Fernando Mesa, José Rodrigo Gonzalez Granada; "Solution to the Klein – Gordon equation using FEM". Journal of Hunan University, **50(7)**, 233-240, (2023).

7. Alemayehu Tamirie Deresse; "Application of Iterative Three-Dimensional Laplace Transform Method for 2-Dimensional Nonlinear Klein-Gordon Equation". Trends in sciences, 20(3): 4410,1-21, (2023).

8. Hamid Mesgarani, Yones Esmaeelzade Aghdam, Ezzatollah Darabi; "A new numerical method for discretization of the nonlinear Klein-Gordon model arising in light waves". Journal of Mathematical Modeling, 12, (1), 71–84, (2024).

9. Chuan-Min He, Lin Li, Shang-Jie Chen; "Nontrivial solution for Klein-Gordon equation coupled with Born-Infeld theory with critical growth". Advances in Nonlinear Analysis; 12: 20220282, (2023).

10. Patel Kaushal, Patel Gautam; “Solution of Klein-Gordon Equation by a Method of Lines Using Reproducing Kernel Hilbert Space Method”. Journal of partial differential equations J. Part. Diff. Eq., 37(3), 251-262 , (2024).

11. Shijie Dong, Zoe Wyatt; “Two dimensional wave-Klein–Gordon equations with a below-critical nonlinearity”. Nonlinear Differ. Equ. Appl., 30(59), (2023).

12. Zhenyu Guo, Xueqian Yan; “Existence of solutions for critical Klein–Gordon equations coupled with Born–Infeld theory in higher dimensions”. Port. Math. 81,1–19, (2024).

13.Akheizer N I.; The classical moment problem , Olivier and Boyd,Edinburgh (1965).

14.Akheizer N I and Krein M G.; “Some questions in the theory of moment”. Am. Math. Soc. Providence, RI (1962).

15.J.A. Shohat and J.D. Tamarkin; The problem of Moments. Mathematic Surveys, Am. Math. Soc., Providence, RI. (1943)

16.D.D. Ang, R. Gorenflo, V.K. Le , D.D. Trong. ; “Moment theory and some inverse problems in potential theory and heat conduction”. Lectures Notes in Mathematics, Springer-Verlag,Berlin (2002)

17.G. Talenti; “Recovering a function from a finite number of moments”. Inverse Problems, **3**, 501-517 (1987).

18.M.B.Pintarelli; “Linear partial differential equations of first order as bi-dimensional inverse moment problema”. Applied Mathematics, **6(6**), 979-989 (2015).

19.M. B. Pintarelli and F. Vericat; “Stability theorem and inversion algorithm for a generalize moment problema”. Far East Journal of Mathematical Sciences, **30**, 253-274 (2008).

20.M.B. Pintarelli and F. Vericat; “Bi-dimensional inverse moment problems”. Far East Journal of Mathematical Sciences, **54**, 1-23 (2011).